

DIVISION OF HOLOMORPHIC FUNCTIONS AND GROWTH CONDITIONS

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ABSTRACT. Let D be a strictly convex domain of \mathbb{C}^n , f_1 and f_2 be two holomorphic functions defined on a neighborhood of \overline{D} and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_l \cap bD$ is transverse for $l = 1$ and $l = 2$, and that $X_1 \cap X_2$ is a complete intersection. We give necessary conditions when $n \geq 2$ and sufficient conditions when $n = 2$ under which a function g to be written as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 in $L^q(D)$, $q \in [1, +\infty)$, or g_1 and g_2 in $BMO(D)$. In order to prove the sufficient condition, we explicitly write down the functions g_1 and g_2 using integral representation formulas and new residue currents.

1. INTRODUCTION

In this article, we are interested in ideals of holomorphic functions and corona type problems. More precisely, if D is a domain of \mathbb{C}^n and f_1, \dots, f_k are k holomorphic functions defined in a neighborhood of \overline{D} , we are looking for condition(s), as close as possible to being necessary and sufficient, under which a function g , holomorphic on D , can be written as

$$(1) \quad g = f_1 g_1 + \dots + f_k g_k,$$

with g_1, \dots, g_k holomorphic on D and satisfying growth conditions at the boundary of D . This kind of problem has been widely studied by many authors under different assumptions.

When D is strictly pseudoconvex and when f_1, \dots, f_k are holomorphic and bounded functions on D , which satisfy $|f|^2 = |f_1|^2 + \dots + |f_k|^2 \geq \delta^2 > 0$, for a given holomorphic and bounded function g , finding functions g_1, \dots, g_k bounded on D is a question known as the Corona Problem. When D is the unit ball of \mathbb{C} , the Corona Problem was solved in 1962 by Carleson in [8]. This question is still open for $n > 1$, even for two generators f_1 and f_2 , and even when D is the unit ball of \mathbb{C}^n .

For $p \in [1, +\infty)$, we denote by $H^p(D)$ the Hardy space of D . When $n > 1$, $k = 2$ and $|f| \geq \delta > 0$, Amar proved in [2] that for any $g \in H^p(D)$, (1) can be solved with g_1 and g_2 in $H^p(D)$. Andersson and Carlsson in [4] generalized this result to any strictly pseudoconvex domain in \mathbb{C}^n and to any $k \geq 2$ and also obtained the BMO -result already announced by Varopoulos in [19]. In [6], they studied the dependence of the g_i 's on the lower bound δ of $|f|$ and they explicitly obtained a constant c_δ such that for all i , $\|g_i\|_{H^p(D)} \leq c_\delta \|g\|_{H^p(D)}$. Of course c_δ goes to infinity when δ goes to 0. In [3], when $|f|$ does not have a positive

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lower bound, Amar and Bruna formulated a sufficient condition in term of the admissible maximum function of $|f|^2 |\log |f||^{2+\varepsilon}$, $\varepsilon > 0$, under which the g_i 's belong to $H^p(D)$.

The corona problem was also studied in the case of the Bergman space $A^p(D)$, the space of holomorphic functions which belong to $L^p(D)$, and in the case of the Zygmund space $\Lambda_\gamma(D)$ by Krantz and Li in [12], and in the case of Hardy-Sobolev spaces by Ortega and Fàbrega in [16].

In the above papers, the first step of the proof in the case of two generators f_1 and f_2 , is to find two smooth functions on D , φ_1 and φ_2 , such that

$$(2) \quad \varphi_1 f_1 + \varphi_2 f_2 = 1;$$

and then to solve the equation

$$(3) \quad \bar{\partial}\varphi = \frac{\overline{f_1} \bar{\partial}\varphi_2 - \overline{f_2} \bar{\partial}\varphi_1}{|f_1|^2 + |f_2|^2}.$$

Then setting $g_1 = g\varphi_1 + \varphi f_2$ and $g_2 = g\varphi_2 - \varphi f_1$, (1) holds and, provided φ belongs to the appropriate space, g_1 and g_2 will belong to $H^p(D)$, $A^p(D)$, \dots . So the problem is reduced to solve the Bezout equation (2) and then to solve the $\bar{\partial}$ -equation (3) with an appropriate regularity.

In [5], Andersson and Carlsson used an alternative technique. They constructed a division formula $g = f_1 T_1(g) + \dots + f_k T_k(g)$ where for all i , T_i is a well chosen Berndtsson-Andersson integral operator, and, still under the assumption $|f| \geq \delta > 0$, they proved that $T_i(g)$ belongs to $H^p(D)$ (resp. $BMO(D)$) when g belongs to $H^p(D)$ (resp. $BMO(D)$). The same kind of technics was also used in [7] by Bonneau, Cumenge and Zeriahi who studied the equation (1) in Lipschitz spaces and in the space $B_M(D) = \{g, \|g\|_{B_M(D)} = \sup_{z \in D} (|g(z)| d(z, bD)^M) < \infty\}$. In this later work, the generators f_1, \dots, f_k may have common zeroes but $\partial f_1 \wedge \dots \wedge \partial f_k$ can not vanish on $bD \cap \{z, f_1(z) = \dots = f_k(z) = 0\}$.

The case of generators having common zeroes has also been investigated by Skoda in [18] for weighted L^2 -spaces. Using and adapting the L^2 -techniques developed by Hörmander, for D pseudoconvex in \mathbb{C}^n , ψ a plurisubharmonic weight on D , f_1, \dots, f_k holomorphic in D , $q = \inf(n, k)$, $\alpha > 1$ and g holomorphic in D such that $\int_D \frac{|g|^2}{|f|^{2\alpha q+2}} e^{-\psi} < \infty$, Skoda showed that there exist $g_1, \dots, g_k \in \mathcal{O}(D)$ such that (1) holds and such that for all i , $\int_D \frac{|g_i|^2}{|f|^{2\alpha q}} e^{-\psi} \leq \frac{\alpha}{\alpha-1} \int_D \frac{|g|^2}{|f|^{2\alpha q+2}} e^{-\psi}$. Moreover the result also holds when k is infinite and there is no restriction on $\partial f_1, \dots, \partial f_k$. However, if one take $g = f_1$ for example, g does not satisfy the assumption of Skoda's theorem in general.

In this article we restrict ourself to a strictly convex domain D of \mathbb{C}^n and we consider the case of two generators f_1 and f_2 , holomorphic in a neighborhood of \bar{D} . We denote by X_1 the set $X_1 = \{z, f_1(z) = 0\}$, and by X_2 the set $X_2 = \{z, f_2(z) = 0\}$. We assume that the intersections $X_1 \cap bD$ and $X_2 \cap bD$ are transverse in the sense of tangent cones and that $X_1 \cap X_2$ is a complete intersection. We seek assumptions on g , holomorphic in D , as close as possible to being necessary and sufficient, under which we can write g as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 holomorphic and belonging to $BMO(D)$ or $L^q(D)$, $q \in [1, +\infty)$.

Let us write D as $D = \{z \in \mathbb{C}^n, \rho(z) < 0\}$ where ρ is a smooth strictly convex function defined on \mathbb{C}^n such that the gradient of ρ does not vanish in a neighborhood \mathcal{U} of bD . We denote by D_r , $r \in \mathbb{R}$, the set $D_r = \{z \in \mathbb{C}^n, \rho(z) < r\}$, by η_ζ the outer unit normal to

$bD_{\rho(\zeta)}$ at a point $\zeta \in \mathcal{U}$ and by v_ζ a smooth unitary complex vector field tangent at ζ to $bD_{\rho(\zeta)}$. As a first result, we show :

Theorem 1.1. *Let D be a strictly convex domain of \mathbb{C}^2 , f_1 and f_2 be two holomorphic functions defined on a neighborhood of \overline{D} and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_l \cap bD$ is transverse for $l = 1$ and $l = 2$, and that $X_1 \cap X_2$ is a complete intersection. Then there exist two integers $k_1, k_2 \geq 1$ depending only from f_1 and f_2 such that if g is any holomorphic function on D which belongs to the ideal generated by f_1 and f_2 and for which there exist two C^∞ smooth functions \tilde{g}_1 and \tilde{g}_2 such that*

- (i) $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$ on D ,
- (ii) there exists $N \in \mathbb{N}$ such that $|\rho|^N \tilde{g}_1$ and $|\rho|^N \tilde{g}_2$ vanish to order k_2 on bD ,
- (iii) there exists $q \in [1, +\infty]$ such that for $l = 1, 2$, $\left| \frac{\partial^{\alpha+\beta} \tilde{g}_l}{\partial \eta_\zeta^\alpha \partial \bar{v}_\zeta^\beta} \right| |\rho|^{\alpha+\frac{\beta}{2}}$ belongs to $L^q(D)$ for all non-negative integers α and β with $\alpha + \beta \leq k_1$,

then there exist two holomorphic functions g_1, g_2 on D which belong to $L^q(D)$ if $q < +\infty$ and to $BMO(D)$ if $q = +\infty$, such that $g_1 f_1 + g_2 f_2 = g$ on D .

The number k_1 and k_2 are almost equal to the maximal order of the singularity of X_1 and X_2 . The functions g_1 and g_2 will be obtained via integral operators acting on \tilde{g}_1 and \tilde{g}_2 . These operators are a combination of a Berndtsson-Andersson kernel and of two (2,2)-currents T_1 and T_2 such that $f_1 T_1 + f_2 T_2 = 1$. So instead of first solving the Bezout equation (2) in the sense of smooth functions, we solve it in the sense of currents and then, instead of solving a $\bar{\partial}$ -equation, we “holomorphy” the smooth solutions \tilde{g}_1 and \tilde{g}_2 of the equation $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$ with integral operators using T_1 and T_2 . These operators can be constructed starting from any currents \tilde{T}_1 and \tilde{T}_2 such that $f_1 \tilde{T}_1 + f_2 \tilde{T}_2 = 1$ (see section 4). However, not all such currents will give operators such that g_1 and g_2 belongs to $L^q(D)$ or $BMO(D)$; they have to be constructed taking into account the behavior of f_1 and f_2 and more precisely the interplay between X_1 and X_2 (see section 3). Moreover, if \tilde{g}_1 and \tilde{g}_2 are already holomorphic and satisfy the assumptions (i) – (iii) of Theorem 1.1, then $g_1 = \tilde{g}_1$ and $g_2 = \tilde{g}_2$.

Observe that in Theorem 1.1, we do not make any assumption on f_1 or f_2 excepted that the intersection $X_1 \cap bD$ and $X_2 \cap bD$ are transverse in the sense of tangent cones, and that $X_1 \cap X_2$ is a complete intersection. This later assumption can be removed provided we add a fourth assumption on \tilde{g}_1 and \tilde{g}_2 . If we moreover assume that

- (iv) $\frac{\partial^{\alpha+\beta} \tilde{g}_1}{\partial \eta_\zeta^\alpha \partial \bar{v}_\zeta^\beta} = 0$ on $X_2 \cap D$ and $\frac{\partial^{\alpha+\beta} \tilde{g}_2}{\partial \eta_\zeta^\alpha \partial \bar{v}_\zeta^\beta} = 0$ on $X_1 \cap D$ for all non negative integers α and β with $\alpha + \beta \leq k_1$,

then Theorem 1.1 also holds whenever $X_1 \cap X_2$ is not complete. However, it then becomes very difficult to find \tilde{g}_1 and \tilde{g}_2 which satisfy this fourth assumption, excepted if $X_1 \cap X_2$ is actually complete.

Indeed, the main difficulty in order to be able to apply Theorem 1.1 is to find the two functions \tilde{g}_1 and \tilde{g}_2 satisfying (i)-(iii). The canonical choice when $|f| \geq \delta > 0$ is to set $\tilde{g}_1 = g \overline{f_1} |f|^{-2}$ and $\tilde{g}_2 = g \overline{f_2} |f|^{-2}$. If $|f| \geq \delta > 0$ and if g belongs to $L^q(D)$, then \tilde{g}_1 and \tilde{g}_2 will satisfy (i)-(iii) and we can then apply Theorem 1.1. However, if $|f|$ does not admit a positive lower bound this will not be necessarily the case. For example, when

$D = \{z \in \mathbb{C}^2, |z_1 - 1|^2 + |z_2|^2 < 1\}$, $f_1(z) = z_2$, $f_2(z) = z_2 - z_1^2$ and $g = f_1$, we can obviously find \tilde{g}_1 and \tilde{g}_2 which satisfy the assumption of Theorem 1.1 but if we make the canonical choices for \tilde{g}_1 and \tilde{g}_2 , they do not fulfill (iii) for $q = \infty$.

Therefore the question of the existence of \tilde{g}_1 and \tilde{g}_2 may itself become a problem that we have to solve. Using first Koranyi balls, we will reduce this global question to a local one and then, using divided differences, we will give numerical conditions under which there indeed exist functions satisfying the hypothesis of Theorem 1.1. We will also prove that these conditions are necessary in order to solve (1) with the g_i 's belonging to $L^q(D)$, $q \in [1, +\infty]$, even in \mathbb{C}^n . This leads us to an effective way of construction of the solutions of (1) belonging to $L^q(D)$ or $BMO(D)$.

The Koranyi balls are defined as follows. We call the coordinates system centered at ζ of basis η_ζ, v_ζ the Koranyi coordinates at ζ . We denote by (z_1^*, z_2^*) the coordinates of a point z in the Koranyi coordinates at ζ . The Koranyi ball centered in ζ of radius r is the set $\mathcal{P}_r(\zeta) := \{\zeta + \lambda\eta_\zeta + \mu v_\zeta, |\lambda| < r, |\mu| < r^{\frac{1}{2}}\}$. The following theorem enables us to go from a local division formula in L^∞ to a global division formula in BMO .

Theorem 1.2. *Let D be a strictly convex domain of \mathbb{C}^2 , f_1 and f_2 be two holomorphic functions defined on a neighborhood of \overline{D} and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_1 \cap bD$ and $X_2 \cap bD$ are transverse, and that $X_1 \cap X_2$ is a complete intersection. Let g be a function holomorphic on D and assume that there exists $\kappa > 0$ such that for all $z \in D$, there exist two functions \hat{g}_1 and \hat{g}_2 , depending on z , C^∞ -smooth on $\mathcal{P}_{\kappa|\rho(z)|}(z)$, such that*

- (a) $g = \hat{g}_1 f_1 + \hat{g}_2 f_2$ on $\mathcal{P}_{\kappa|\rho(z)|}(z)$;
- (b) for all non negative integers $\alpha, \beta, \bar{\alpha}$ and $\bar{\beta}$, there exist $c > 0$, not depending on z , such that $\sup_{\mathcal{P}_{\kappa|\rho(z)|}(z)} \left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \hat{g}_l}{\partial z_1^*{}^\alpha \partial z_2^*{}^\beta \partial \bar{z}_1^*{}^{\bar{\alpha}} \partial \bar{z}_2^*{}^{\bar{\beta}}} \right| \leq c$ for $l = 1$ and $l = 2$.

Then there exist two smooth functions \tilde{g}_1 and \tilde{g}_2 which satisfy the assumptions (i)-(iii) of Theorem 1.1 for $q = +\infty$.

An analogous theorem holds true in the L^q -case (see Theorem 6.1). We observe that if, for all $z \in D$, there exist two functions \hat{g}_1 and \hat{g}_2 , holomorphic and bounded on $\mathcal{P}_{2\kappa|\rho(z)|}(z)$ by a constant c which does not depend from z , and such that $g = \hat{g}_1 f_1 + \hat{g}_2 f_2$ on $\mathcal{P}_{2\kappa|\rho(z)|}(z)$, then Cauchy's inequalities implies that \hat{g}_1 and \hat{g}_2 satisfy the assumption of Theorem 1.2 on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ for all z . Therefore Theorem 1.2 implies that the global solvability of (1) in the BMO space of D is nearly equivalent to its uniform local solvability. In order to prove Theorem 1.2, we will cover D with Koranyi balls and using a suitable partition of unity, we will glue together the \hat{g}_1 and \hat{g}_2 which we got on each ball. We point out that when we glue together the local \hat{g}_1 's, excepted if $X_1 \cap X_2$ is a complete intersection, in general the "fourth" assumption (iv) of Theorem 1.1 is not satisfied. This is why we chose to present Theorem 1.1 as we did.

When looking for necessary conditions in order to solve (1) with g_1 and g_2 bounded, we first observe that g is trivially bounded by $\max(\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty})(|f_1| + |f_2|)$. Therefore, in order for g to be written as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 bounded, it is necessary that $\frac{|g|}{|f_1| + |f_2|}$ be bounded. However this condition alone is not sufficient in general. Consider for

example the ball $D := \{z \in \mathbb{C}^2, \rho(z) = |z_1 - 1|^2 + |z_2|^2 - 1 < 0\}$, $f_1(z) = z_2^2$, $f_2(z) = z_2^2 - z_1^q$ and $g(z) = z_1^{\frac{q}{2}} z_2$ where $q \geq 3$ is an odd integer. Then $g(z) = z_2 z_1^{-\frac{q}{2}} f_1(z) - z_2 z_1^{-\frac{q}{2}} f_2(z)$, so g belongs to the ideal generated by f_1 and f_2 , and $\frac{|g|}{|f_1| + |f_2|}$ is bounded on D by $\frac{3}{2}$; in particular, the classical choice $\tilde{g}_1 = \frac{g f_1}{|f_1|^2 + |f_2|^2}$ and $\tilde{g}_2 = \frac{g f_2}{|f_1|^2 + |f_2|^2}$ are smooth and bounded on D . However, (1) can not be solved with g_1 and g_2 bounded on D . In order to see this, a good tool is divided differences. Indeed, on the one hand, if $g = g_1 f_1 + g_2 f_2$, then $g_1 = g \cdot f_1^{-1}$ on $X_2 \setminus X_1$. On the other hand, if g_1 is bounded, for all $z \in D$, all unit vector v tangent to $bD_{-\rho(z)}$ at z , all complex numbers λ_1 and λ_2 with $\rho(z + \lambda_1 v) < \frac{1}{2}\rho(z)$ and $\rho(z + \lambda_2 v) < \frac{1}{2}\rho(z)$, the divided difference $\frac{g_1(z + \lambda_1 v) - g_1(z + \lambda_2 v)}{\lambda_1 - \lambda_2}$ behaves like the derivative $\frac{\partial g_1}{\partial v}$ at some point $z + \mu v$ where μ is an element of the segment $[\lambda_1, \lambda_2]$ (see [17]). Cauchy's inequalities then imply that, up to a uniform multiplicative constant, $\frac{g_1(z + \lambda_1 v) - g_1(z + \lambda_2 v)}{\lambda_1 - \lambda_2}$ is bounded by $\|g_1\|_{L^\infty(D)} |\rho(z)|^{-\frac{1}{2}}$.

So when we compute the divided differences of g_1 at points $z + \lambda_1 v$ and $z + \lambda_2 v$ which belong to $X_2 \setminus X_1$, whatever g_1 and g_2 may be, we actually compute the divided difference of $g \cdot f_1^{-1}$; if g_1 is bounded, this divided difference times $|\rho(z)|^{\frac{1}{2}}$ must be bounded by some uniform constant. But in our example, this is not the case because for small $\varepsilon > 0$, setting $z = (\varepsilon, 0)$, $v = (0, 1)$, $\lambda_1 = \varepsilon^{\frac{q}{2}}$ and $\lambda_2 = -\varepsilon^{\frac{q}{2}}$, we have that $\frac{(g \cdot f_1^{-1})(z + \lambda_1 v) - (g \cdot f_1^{-1})(z + \lambda_2 v)}{\lambda_1 - \lambda_2} |\rho(z)|^{\frac{1}{2}} = \varepsilon^{\frac{1-q}{2}}$ which is unbounded when ε goes to zero.

In \mathbb{C}^n , we will prove that the divided differences of any order of $g \cdot f_1^{-1}$ and $g \cdot f_2^{-1}$ must satisfy some boundedness properties when (1) is solvable with g_1 and g_2 in $L^q(D)$, $q \in [1, +\infty]$ (see Theorems 6.3 and 6.5 for precise statements). Conversely, in \mathbb{C}^2 , if those boundedness properties are satisfied, up to an error term we will be able to construct by interpolation \hat{g}_1 and \hat{g}_2 on any Koranyi balls which satisfy the assumptions of Theorem 1.2; applying Theorem 1.2, we will then prove that there exist two functions g_1 and g_2 holomorphic on D , belonging to $BMO(D)$ or $L^q(D)$, $q \in [1, +\infty)$, such that $g = g_1 f_1 + g_2 f_2$ (see Theorem 6.4 and 6.6).

The article is organized as follows. In Section 2, we recall some tools needed for the construction and the estimation of the division formula. In Section 3, we construct the currents which enable us to construct our division formula in Section 4. In Section 5 we establish Theorem 1.1 and finally, in Section 6, we prove the theorems related to local division in the L^∞ and L^q case.

2. NOTATIONS AND TOOLS

2.1. Koranyi balls. The Koranyi balls centered at a point z in D have properties linked with distance from z to the boundary of D in a direction v . For $z \in \mathbb{C}^n$, v a unit vector in \mathbb{C}^n , and $\varepsilon > 0$, the distance from z to $bD_{\rho(z)+\varepsilon}$ in the direction v is defined by

$$\tau(z, v, \varepsilon) = \sup\{\tau > 0, \rho(z + \lambda v) - \rho(z) < \varepsilon \text{ for all } \lambda \in \mathbb{C}, |\lambda| < \tau\}.$$

Thus $\tau(z, v, \varepsilon)$ is the maximal radius $r > 0$ such that the disc $\Delta_{z,v}(r) = \{z + \lambda v, |\lambda| < r\}$ is in $D_{\rho(z)+\varepsilon}$; if v is a tangent vector to $bD_{\rho(z)}$ at z , then $\tau(z, v, \varepsilon)$ is comparable to $\varepsilon^{\frac{1}{2}}$ and $\tau(z, \eta_z, \varepsilon)$ is comparable to ε .

Before we recall the properties of the Koranyi balls we will need, we adopt the following

notation. We write $A \lesssim B$ if there exists some constant $c > 0$ such that $A \leq cB$. Each time we will mention from which parameters c depends. We will write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$ both holds.

Proposition 2.1. *There exists a neighborhood \mathcal{U} of bD and positive real numbers κ and c_1 such that*

- (i) *for all $\zeta \in \mathcal{U} \cap D$, $\mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta)$ is included in D .*
- (ii) *for all $\varepsilon > 0$, all $\zeta, z \in \mathcal{U}$, $\mathcal{P}_\varepsilon(\zeta) \cap \mathcal{P}_\varepsilon(z) \neq \emptyset$ implies $\mathcal{P}_\varepsilon(z) \subset \mathcal{P}_{c_1\varepsilon}(\zeta)$.*
- (iii) *for all $\varepsilon > 0$ sufficiently small, all $z \in \mathcal{U}$, all $\zeta \in \mathcal{P}_\varepsilon(z)$ we have $|\rho(z) - \rho(\zeta)| \leq c_1\varepsilon$.*
- (iv) *for all $\varepsilon > 0$, all unit vectors $v \in \mathbb{C}^n$, all $z \in \mathcal{U}$ and all $\zeta \in \mathcal{P}_\varepsilon(z)$, $\tau(z, v, \varepsilon) \approx \tau(\zeta, v, \varepsilon)$ uniformly with respect to ε, z and ζ .*

For \mathcal{U} given by Proposition 2.1 and z and ζ belonging to \mathcal{U} , we set $\delta(z, \zeta) = \inf\{\varepsilon > 0, \zeta \in \mathcal{P}_\varepsilon(z)\}$. Proposition 2.1 implies that δ is a pseudo-distance in the following sense:

Proposition 2.2. *For \mathcal{U} and c_1 given by Proposition 2.1 and for all z, ζ and ξ belonging to \mathcal{U} we have*

$$\frac{1}{c_1}\delta(\zeta, z) \leq \delta(z, \zeta) \leq c_1\delta(\zeta, z)$$

and

$$\delta(z, \zeta) \leq c_1(\delta(z, \xi) + \delta(\xi, \zeta))$$

2.2. Berndtsson-Andersson reproducing kernel. Berndtsson-Andersson's kernel will be one of our most important ingredients in the construction of the functions g_1 and g_2 of Theorem 1.1. We now recall its definition for D a strictly convex domain of \mathbb{C}^2 . We set $h_1(\zeta, z) = -\frac{\partial \rho}{\partial \zeta_1}(\zeta)$, $h_2(\zeta, z) = -\frac{\partial \rho}{\partial \zeta_2}(\zeta)$, $h = \sum_{i=1,2} h_i d\zeta_i$ and $\tilde{h} = \frac{1}{\rho}h$. For a $(1,0)$ -form $\beta(\zeta, z) = \sum_{i=1,2} \beta_i(\zeta, z) d\zeta_i$ we set $\langle \beta(\zeta, z), \zeta - z \rangle = \sum_{i=1,2} \beta_i(\zeta, z)(\zeta_i - z_i)$. Then we define the Berndtsson-Andersson reproducing kernel by setting for an arbitrary positive integer N , $n = 1, 2$ and all $\zeta, z \in D$:

$$P^{N,n}(\zeta, z) = C_{N,n} \left(\frac{1}{1 + \langle \tilde{h}(\zeta, z), \zeta - z \rangle} \right)^{N+n} (\bar{\partial} \tilde{h})^n,$$

where $C_{N,n} \in \mathbb{C}$ is a suitable constant. We also set $P^{N,n}(\zeta, z) = 0$ for all $z \in D$ and all $\zeta \notin D$. Then the following theorem holds true (see [9]):

Theorem 2.3. *For all $g \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ we have*

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z).$$

In order to find an upper bound for this kernel, we will have to write h in the Koranyi coordinates at some point ζ_0 belonging to D . We set $h_1^* = -\frac{\partial \rho}{\partial \zeta_1^*}(\zeta)$ and $h_2^* = -\frac{\partial \rho}{\partial \zeta_2^*}(\zeta)$. Then h is equal to $\sum_{i=1,2} h_i^* d\zeta_i^*$ and satisfies the following Proposition.

Proposition 2.4. *There exists a neighborhood \mathcal{U} of bD such that for all $\zeta \in D \cap \mathcal{U}$, all $\varepsilon > 0$ sufficiently small and all $z \in \mathcal{P}_\varepsilon(\zeta)$ we have*

- (i) $|\rho(\zeta) + \langle h(\zeta, z), \zeta - z \rangle| \gtrsim \varepsilon + |\rho(\zeta)| + |\rho(z)|$,
- (ii) $|h_1^*(\zeta, z)| \lesssim 1$,
- (iii) $|h_2^*(\zeta, z)| \lesssim \varepsilon^{\frac{1}{2}}$,

and there exists $c > 0$ not depending from ζ nor from ε such that for all $z \in \mathcal{P}_\varepsilon(\zeta) \setminus c\mathcal{P}_\varepsilon(\zeta)$ we have

$$|\langle h(\zeta, z), \zeta - z \rangle| \gtrsim \varepsilon + |\rho(z)| + |\rho(\zeta)|,$$

uniformly with respect to ζ, z and ε .

3. CONSTRUCTION OF THE CURRENTS

In [15], the following was proved : If f_1 and f_2 are two holomorphic functions near the origin in \mathbb{C}^n , two currents T and S such that $f_1 T = 1$, $f_2 S = \bar{\partial} T$ and $f_1 S = 0$ were constructed on a sufficiently small neighborhood \mathcal{U} of 0. It was also proved that if T and S are any currents satisfying these three hypothesis, then any function g holomorphic on \mathcal{U} can be written as $g = f_1 g_1 + f_2 g_2$ on \mathcal{U} if and only if $g \bar{\partial} S = 0$. Moreover, g_1 and g_2 can be explicitly written down using T and S .

Here, when f_1 and f_2 are holomorphic on a domain D , we first want to obtain a decomposition $g = g_1 f_1 + g_2 f_2$ on the whole domain D and then secondly we want to obtain growth estimates on g_1 and g_2 . As a first approach, we could try to globalize the currents T and S of [15] in order to have a global decomposition. However, such an approach would fail to give the growth estimates we want.

In [15], f_1 plays a leading role and T is constructed independently of f_2 , using only f_1 . Then S is constructed using f_1 and f_2 . If we assume for example that f_1 vanishes at a point ζ_0 near bD , because T is constructed independently of f_2 , it seems difficult to prove that g_1 is bounded excepted if we require that g vanishes at ζ_0 too. However, considering $g = f_2$, we easily see that in general this condition is not necessary when one wants to write g as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 bounded for example. So the current in [15] probably does not give a good decomposition.

Actually, it appears that f_2 must be prioritized in the construction of the currents near a boundary point ζ_0 such that $f_1(\zeta_0) = 0$ and $f_2(\zeta_0) \neq 0$ or more generally when f_2 is in some sense greater than f_1 and conversely. Following this idea, we construct two currents T_1 and T_2 such that $f_1 T_1 + f_2 T_2 = 1$ on D . These currents are defined locally and using a suitable partition of unity we glue together the local currents and get a global current.

Let ε_0 be a small positive real number to be chosen later and let ζ_0 be a point in \bar{D} . We distinguish three cases.

If ζ_0 belongs to $D_{-\varepsilon_0}$, we do not need to be careful. Using Weierstrass' preparation theorem when ζ_0 belongs to X_1 , we write $f_1 = u_{0,1} P_{0,1}$ where $u_{0,1}$ is a non vanishing holomorphic function in a neighborhood $\mathcal{U}_0 \subset D_{-\frac{\varepsilon_0}{2}}$ of ζ_0 and $P_{0,1}(\zeta) = \zeta_2^{i_{0,1}} + \zeta_2^{i_{0,1}-1} a_{0,1}^{(1)}(\zeta_1) + \dots + a_{0,1}^{(i_{0,1})}(\zeta_1)$, $a_{0,1}^{(k)}$ holomorphic on \mathcal{U}_0 for all k . If ζ_0 does not belong to X_1 , we set $P_{0,1} = 1$, $i_{0,1} = 0$, $u_{0,1} = f_1$ and we still have $f_1 = u_{0,1} P_{0,1}$ with $u_{0,1}$ which does not vanish on some neighborhood \mathcal{U}_0 of ζ_0 .

For a smooth $(2,2)$ -form φ compactly supported in \mathcal{U}_0 we set

$$\begin{aligned} \langle T_{0,1}, \varphi \rangle &= \frac{1}{c_0} \int_{\mathcal{U}_0} \frac{\overline{P_1(\zeta)}}{f_1(\zeta)} \frac{\partial^{i_{0,1}} \varphi}{\partial \bar{\zeta}_2^{i_{0,1}}}(\zeta), \\ \langle T_{0,2}, \varphi \rangle &= 0, \end{aligned}$$

where c_0 is a suitable constant (see [15]). Integrating by parts we get $f_1 T_{0,1} + f_2 T_{0,2} = 1$ on \mathcal{U}_0 .

If ζ_0 belongs to $bD \setminus (X_1 \cap X_2)$, without restriction we assume that $f_1(\zeta_0) \neq 0$. Let \mathcal{U}_0 be a neighborhood of ζ_0 such that f_1 does not vanish in \mathcal{U}_0 . As in the previous case when $f_1(\zeta_0) \neq 0$, we set $P_{0,1} = 1$, $i_{0,1} = 0$, $u_{0,1} = f_1$ and for any smooth $(2, 2)$ -form φ compactly supported in $D \cap \mathcal{U}_0$ we put

$$\begin{aligned}\langle T_{0,1}, \varphi \rangle &= \frac{1}{c_0} \int_{\mathcal{U}_0} \frac{\overline{P_1(\zeta)}}{f_1(\zeta)} \frac{\partial^{i_{0,1}} \varphi}{\partial \overline{\zeta_2^{i_{0,1}}}}(\zeta), \\ \langle T_{0,2}, \varphi \rangle &= 0.\end{aligned}$$

where as previously c_0 is a suitable constant. Again, we have $f_1 T_{0,1} + f_2 T_{0,2} = 1$ on $\mathcal{U}_0 \cap D$.

If ζ_0 belongs to $X_1 \cap X_2 \cap bD$, as in [1], we cover a neighborhood \mathcal{U}_0 of ζ_0 by a family of polydiscs $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, $j \in \mathbb{N}$ and $k \in \{1, \dots, n_j\}$ such that :

- (i) For all $j \in \mathbb{N}$, and all $k \in \{1, \dots, n_j\}$, $z_{j,k}$ belongs to $bD_{-(1-c\kappa)^j \varepsilon_0}$.
- (ii) For all $j \in \mathbb{N}$, all $k, l \in \{1, \dots, n_j\}$, $k \neq l$, we have $\delta(z_{j,k}, z_{j,l}) \geq c\kappa(1 - c\kappa)^j \varepsilon_0$.
- (iii) For all $j \in \mathbb{N}$, all $z \in bD_{-(1-c\kappa)^j \varepsilon_0}$, there exists $k \in \{1, \dots, n_j\}$ such that $\delta(z, z_{j,k}) < c\kappa(1 - c\kappa)^j \varepsilon_0$,
- (iv) $D \cap \mathcal{U}_0$ is included in $\cup_{j=0}^{+\infty} \cup_{k=1}^{n_j} \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$,
- (v) there exists $M \in \mathbb{N}$ such that for $z \in D \setminus D_{-\varepsilon_0}$, $\mathcal{P}_{4\kappa|\rho(z)|}(z)$ intersect at most M Koranyi balls $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$.

Such a family of polydiscs will be called a κ -covering.

We define on each polydisc $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ two currents $T_{0,1}^{(j,k)}$ and $T_{0,2}^{(j,k)}$ such that $f_1 T_{0,1}^{(j,k)} + f_2 T_{0,2}^{(j,k)} = 1$ as follows. We denote by $\Delta_\xi(\varepsilon)$ the disc of center ξ and radius ε and by $(\zeta_{0,1}^*, \zeta_{0,2}^*)$ the coordinates of ζ_0 in the Koranyi basis at $z_{j,k}$. In [1] were proved the next two propositions :

Proposition 3.1. *If $\kappa > 0$ is small enough and if $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l \neq \emptyset$ then $|\zeta_{0,1}^*| \geq 2\kappa|\rho(z_{j,k})|$.*

We assume κ so small that Proposition 3.1 holds for both X_1 and X_2 with the same κ . When $|\zeta_{0,1}^*| \geq 2\kappa|\rho(z_{j,k})|$ then X_l can be parametrized as follows (see [1]) :

Proposition 3.2. *If $|\zeta_{0,1}^*| \geq 2\kappa|\rho(z_{j,k})|$, for $l = 1$ and $l = 2$, there exists p_l functions $\alpha_{l,1}^{(j,k)}, \dots, \alpha_{l,p_l}^{(j,k)}$ holomorphic on $\Delta_0(2\kappa|\rho(z_{j,k})|)$, there exists $r > 0$, not depending from j nor from k , and there exists $u_l^{(j,k)}$ holomorphic on the ball of center ζ_0 and radius r such that :*

- (i) $\frac{\partial \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^*}$ is bounded on $\Delta_0(2\kappa|\rho(z_{j,k})|)$ uniformly with respect to j and k ,
- (ii) for all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$, $f_l(\zeta) = u_l^{(j,k)}(\zeta) \prod_{i=1}^{p_l} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*))$.

Now we define $T_{0,1}^{(j,k)}$ and $T_{0,2}^{(j,k)}$ with the following settings.
 If $|z_{0,1}^*| < 2\kappa|\rho(z_{j,k})|$ we set for $l = 1$ and $l = 2$:

$$\begin{aligned} I_l^{(j,k)} &:= \emptyset; \\ i_l^{(j,k)} &:= 0; \\ P_l^{(j,k)}(\zeta) &:= 1. \end{aligned}$$

If $|z_{0,1}^*| \geq 2\kappa|\rho(z_{j,k})|$ we set for $l = 1$ and $l = 2$:

$$\begin{aligned} I_l^{(j,k)} &:= \{i, \exists z_1^* \in \mathbb{C}, |z_1^*| < \kappa|\rho(z_{j,k})| \text{ and } |\alpha_{l,i}^{(j,k)}(z_1^*)| < (2\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}\}; \\ i_l^{(j,k)} &:= \#I_l^{(j,k)}, \text{ the cardinal of } I_l^{(j,k)}; \\ P_l^{(j,k)}(\zeta) &:= \prod_{i \in I_l^{(j,k)}} \left(\zeta_2^* - \alpha_{i,l}^{(j,k)}(\zeta_1^*) \right). \end{aligned}$$

In both case we set

$$\begin{aligned} \mathcal{U}_1^{(j,k)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \left| \frac{f_1(\zeta)\rho(z_{j,k})^{i_1^{(j,k)}}}{P_1^{(j,k)}(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)\rho(z_{j,k})^{i_2^{(j,k)}}}{P_2^{(j,k)}(\zeta)} \right| \right\}, \\ \mathcal{U}_2^{(j,k)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \frac{2}{3} \left| \frac{f_2(\zeta)\rho(z_{j,k})^{i_2^{(j,k)}}}{P_2^{(j,k)}(\zeta)} \right| > \left| \frac{f_1(\zeta)\rho(z_{j,k})^{i_1^{(j,k)}}}{P_1^{(j,k)}(\zeta)} \right| \right\}, \end{aligned}$$

so that $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}) = \mathcal{U}_1^{(j,k)} \cup \mathcal{U}_2^{(j,k)}$.

For $l = 1, 2$ and for a smooth $(2, 2)$ -form φ compactly supported in $\mathcal{U}_l^{(j,k)}$ we set

$$\langle T_{0,l}^{(j,k)}, \varphi \rangle := \int_{\mathbb{C}^2} \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} \frac{\partial^{i_l^{(j,k)}} \varphi}{\partial \zeta_2^{* i_l^{(j,k)}}}(\zeta).$$

Integrating $i_l^{(j,k)}$ -times by parts, we get $f_l T_{0,l}^{(j,k)} = c_l^{(j,k)}$ on $\mathcal{U}_l^{(j,k)}$ where $c_l^{(j,k)}$ is an integer bounded by $i_l^{(j,k)}!$ (see [15]).

Now we glue together the currents $T_{0,l}^{(j,k)}$ in order to define the current $T_{0,l}$, $l = 1, 2$, such that $f_1 T_{0,1} + f_2 T_{0,2} = 1$ on $D \cap \mathcal{U}_0$. Let $(\tilde{\chi}_{j,k})_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}}$ be a partition of unity subordinated to the covering $(\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}))_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}}$ of \mathcal{U}_0 . Without restriction, we

assume that $\left| \frac{\partial^{\alpha+\beta+\bar{\alpha}+\bar{\beta}} \tilde{\chi}_{j,k}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \zeta_1^{\bar{\alpha}} \partial \zeta_2^{\bar{\beta}}}(\zeta) \right| \lesssim \frac{1}{|\rho(z_{j,k})|^{\alpha+\bar{\alpha}+\frac{\beta+\bar{\beta}}{2}}}$. Let also χ be a smooth function on $\mathbb{C} \setminus \{0\}$ such that $\chi(z_1, z_2) = 1$ if $|z_1| > \frac{2}{3}|z_2|$ and $\chi(z_1, z_2) = 0$ if $|z_1| < \frac{1}{3}|z_2|$ and let us define

$$\begin{aligned} \chi_1^{(j,k)}(\zeta) &= \tilde{\chi}_{j,k}(\zeta) \cdot \chi \left(\frac{f_1(\zeta)\rho(z_{j,k})^{i_1^{(j,k)}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)\rho(z_{j,k})^{i_2^{(j,k)}}}{P_2^{(j,k)}(\zeta)} \right), \\ \chi_2^{(j,k)}(\zeta) &= \tilde{\chi}_{j,k}(\zeta) \cdot \left(1 - \chi \left(\frac{f_1(\zeta)\rho(z_{j,k})^{i_1^{(j,k)}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)\rho(z_{j,k})^{i_2^{(j,k)}}}{P_2^{(j,k)}(\zeta)} \right) \right). \end{aligned}$$

For $l = 1$ and $l = 2$, the support of $\chi_l^{(j,k)}$ is included in $\mathcal{U}_l^{(j,k)}$ so we can put

$$T_{0,l} = \sum_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}} \frac{1}{c_l^{(j,k)}} \chi_l^{(j,k)} T_{0,l}^{(j,k)}$$

and we have $f_1 T_{0,1} + f_2 T_{0,2} = 1$ on $\mathcal{U}_0 \cap D$.

Now for all $\zeta_0 \in bD \cup \overline{D_{-\varepsilon_0}}$ we have constructed a neighborhood \mathcal{U}_0 of ζ_0 and two currents $T_{0,1}$ and $T_{0,2}$ such that $f_1 T_{0,1} + f_2 T_{0,2} = 1$ on $\mathcal{U}_0 \cap D$. If $\varepsilon_0 > 0$ is sufficiently small, we can cover \overline{D} by finitely many open sets $\mathcal{U}_1, \dots, \mathcal{U}_n$. Let χ_1, \dots, χ_n be a partition of unity subordinated to this family of open sets and $T_{1,1}, \dots, T_{n,1}$ and $T_{1,2}, \dots, T_{n,2}$ be the corresponding currents defined on $\mathcal{U}_1, \dots, \mathcal{U}_n$. We glue together this current and we set

$$T_1 = \sum_{j=1}^k \chi_j T_{j,1} \text{ and } T_2 = \sum_{j=1}^n \chi_j T_{j,2},$$

so that $f_1 T_1 + f_2 T_2 = 1$ on D . Moreover T_1 and T_2 are currents supported in \overline{D} thus they have a finite order k_2 and we can apply T_1 and T_2 to function of class C^{k_2} with support in \overline{D} . This gives k_2 from Theorem 1.1.

4. THE DIVISION FORMULA

In this part, given any two currents T_1 and T_2 of order k_2 such that $f_1 T_1 + f_2 T_2 = 1$, assuming that g is a holomorphic function on D which belonging to the ideal generated by f_1 and f_2 , and which can be written as $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$, where \tilde{g}_1 and \tilde{g}_2 are two C^∞ -smooth functions on D such that $|\rho|^N \tilde{g}_1$ and $|\rho|^N \tilde{g}_2$ vanish to order k_2 on bD for some $N \in \mathbb{N}$ sufficiently big, we write g as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 holomorphic on D . We point out that the formula we will get is valid for any T_1 and T_2 of order k_2 such that $f_1 T_1 + f_2 T_2 = 1$.

Under our assumptions, for $k = 1$ and $k = 2$ and all fixed $z \in D$, $\tilde{g}_1 P^{N,k}(\cdot, z)$ and $\tilde{g}_2 P^{N,k}(\cdot, z)$ can be extended by zero outside D and are of class C^{k_2} on \mathbb{C} . So we can apply T_1 and T_2 to $\tilde{g}_1 P^{N,k}(\cdot, z)$ and $\tilde{g}_2 P^{N,k}(\cdot, z)$. Now we construct a division formula.

For $l = 1, 2$, we denote by $b_l = b_{l,1} d\zeta_1 + b_{l,2} d\zeta_2$ a $(1, 0)$ -form such that $f_l(z) - f_l(\zeta) = \sum_{i=1,2} b_{l,i}(\zeta, z)(z_i - \zeta_i)$. For the estimates, we will take $b_{l,i}(\zeta, z) = \int_0^1 \frac{\partial f_l}{\partial \zeta_i}(\zeta + t(z - \zeta)) dt$, but this is not necessary to get a division formula.

From Theorem 2.3, we have for all $z \in D$:

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z)$$

and since $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$

$$\begin{aligned} g(z) &= f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) + f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) \\ (4) \quad &+ \int_D \tilde{g}_1(\zeta) (f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) + \int_D \tilde{g}_2(\zeta) (f_2(\zeta) - f_2(z)) P^{N,2}(\zeta, z). \end{aligned}$$

Now from [14], Lemma 3.4, there exists $\tilde{c}_{N,2}$ such that

$$(f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) = \tilde{c}_{N,2} b_1(\zeta, z) \wedge \bar{\partial} P^{N,1}(\zeta, z)$$

and since by assumption $\tilde{g}_1 P^{N,1}$ vanishes on bD , Stokes' Theorem yields

$$(5) \quad \int_D \tilde{g}_1(\zeta) (f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) = \tilde{c}_{N,2} \int_D \bar{\partial} \tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z).$$

We now use the fact that $f_1 T_1 + f_2 T_2 = 1$ in order to rewrite the former integral :

$$\begin{aligned} & \int_D \bar{\partial} \tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z) \\ &= \langle f_1 T_1 + f_2 T_2, \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ &= \langle f_1 T_1, \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + f_2(z) \langle T_2, \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ (6) \quad & + \langle T_2, (f_2 - f_2(z)) \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle. \end{aligned}$$

Again from [14], Lemma 3.4, there exists $\tilde{c}_{N,1}$ such that

$$\begin{aligned} & (f_2(\zeta) - f_2(z)) b_1(\zeta, z) \wedge P^{N,1}(\zeta, z) - (f_1(\zeta) - f_1(z)) b_2(\zeta, z) \wedge P^{N,1}(\zeta, z) \\ &= \tilde{c}_{N,1} b_1(\zeta, z) \wedge b_2(\zeta, z) \wedge \bar{\partial} P^{N,0}(\zeta, z). \end{aligned}$$

So

$$\begin{aligned} & \langle T_2, (f_2 - f_2(z)) \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ &= -f_1(z) \langle T_2, \bar{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \langle T_2, f_1 \bar{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ (7) \quad & + \tilde{c}_{N,1} \langle T_2, \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \bar{\partial} P^{N,0}(\cdot, z) \rangle \end{aligned}$$

We plug together (5), (6) and (7) and their analogue for $\int_D g_2(\zeta) (f_2(\zeta) - f_2(z)) P^{N,2}(\zeta, z)$ in (4) and we get

$$\begin{aligned} g(z) &= f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_1(z) \langle T_2, \bar{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ &+ \tilde{c}_{N,2} f_2(z) \langle T_2, \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ &+ f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_2(z) \langle T_1, \bar{\partial} \tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ &+ \tilde{c}_{N,2} f_1(z) \langle T_1, \bar{\partial} \tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ (8) \quad & + \tilde{c}_{N,2} \langle T_1, f_1 \bar{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \tilde{c}_{N,2} \langle T_2, f_1 \bar{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ (9) \quad & + \tilde{c}_{N,2} \langle T_2, f_2 \bar{\partial} \tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \tilde{c}_{N,2} \langle T_1, f_2 \bar{\partial} \tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\ & + \tilde{c}_{N,2} \tilde{c}_{N,1} \langle \bar{\partial} \tilde{g}_1 \wedge T_2 - \bar{\partial} \tilde{g}_2 \wedge T_1, b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \bar{\partial} P^{N,0}(\cdot, z) \rangle \end{aligned}$$

Now since $\bar{\partial} g = f_1 \bar{\partial} \tilde{g}_1 + f_2 \bar{\partial} \tilde{g}_2 = 0$, the line (8) and (9) vanish. Therefore in order to get our division formula, it suffices to prove that $\bar{\partial}(\bar{\partial} \tilde{g}_1 \wedge T_2 - \bar{\partial} \tilde{g}_2 \wedge T_1) = 0$.

When $X_1 \cap X_2$ is not a complete intersection and when assumption (iv) in the introduction is satisfied by \tilde{g}_1 and \tilde{g}_2 , one can prove that $\bar{\partial} \tilde{g}_1 \wedge \bar{\partial} T_2 = 0$ and $\bar{\partial} \tilde{g}_2 \wedge \bar{\partial} T_1 = 0$.

When $X_1 \cap X_2$ is a complete intersection, we prove that for any $\zeta_0 \in D$ there exists a neighborhood \mathcal{U}_0 of ζ_0 such that for all $(2, 1)$ -form φ , smooth and supported in \mathcal{U}_0 , we have $\langle \bar{\partial} \tilde{g}_1 \wedge T_2 - \bar{\partial} \tilde{g}_2 \wedge T_1, \bar{\partial} \varphi \rangle = 0$.

Let ζ_0 be a point in D . By assumption on g , there exists a neighborhood \mathcal{U}_0 of ζ_0 and two holomorphic functions γ_1 and γ_2 such that $g = \gamma_1 f_1 + \gamma_2 f_2$ on \mathcal{U}_0 . We now use the following lemma from which we postpone the proof to the end of this section :

Lemma 4.1. *Let f_1 and f_2 be two holomorphic functions defined in a neighborhood of 0 in \mathbb{C}^2 , $X_1 = \{z, f_1(z) = 0\}$ and $X_2 = \{z, f_2(z) = 0\}$. We assume that $X_1 \cap X_2$ is a complete intersection and that 0 belongs to $X_1 \cap X_2$. Let φ_1 and φ_2 be two C^∞ -smooth functions such that $f_1\varphi_1 = f_2\varphi_2$.*

Then, $\frac{\varphi_1}{f_2}$ and $\frac{\varphi_2}{f_1}$ are C^∞ -smooth in a neighborhood of 0.

Lemma 4.1 implies that the function $\psi = \frac{\tilde{g}_1 - \gamma_1}{f_2} = \frac{\gamma_2 - \tilde{g}_2}{f_1}$ is smooth on a perhaps smaller neighborhood of ζ_0 still denoted by \mathcal{U}_0 . Thus

$$\begin{aligned} \langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, \bar{\partial}\varphi \rangle &= \langle \bar{\partial}(\tilde{g}_1 - \gamma_1) \wedge T_2 + \bar{\partial}(\gamma_2 - \tilde{g}_2) \wedge T_1, \bar{\partial}\varphi \rangle \\ &= \langle \bar{\partial}(f_2\psi) \wedge T_2 + \bar{\partial}(f_1\psi) \wedge T_1, \bar{\partial}\varphi \rangle \\ &= \langle f_2T_2 + f_1T_1, \bar{\partial}\psi \wedge \bar{\partial}\varphi \rangle \\ &= \int_{\mathcal{U}_0} \bar{\partial}\psi \wedge \bar{\partial}\varphi \end{aligned}$$

and since φ is supported in \mathcal{U}_0 we have $\int_{\mathcal{U}_0} \bar{\partial}\psi \wedge \bar{\partial}\varphi = - \int_{\mathcal{U}_0} d(\varphi \bar{\partial}\psi) = 0$ and so

$$\langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, \bar{\partial}\varphi \rangle = 0.$$

Now we set

$$\begin{aligned} g_1(z) &= \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) \\ &\quad + \tilde{c}_{N,2} (\langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle - \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle) \\ g_2(z) &= \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) \\ &\quad + \tilde{c}_{N,2} (\langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle - \langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle) \end{aligned}$$

and we have

$$g = g_1 f_1 + g_2 f_2$$

with g_1 and g_2 holomorphic on D . We notice that if \tilde{g}_1 and \tilde{g}_2 are already holomorphic functions then $g_1 = \tilde{g}_1$ and $g_2 = \tilde{g}_2$.

Proof of Lemma 4.1 : Maybe after a unitary change of coordinates, we can assume that for $l = 1, 2$, the function f_l is given by $f_l(z, w) = z^{k_l} + a_1^{(l)}(w)z^{k_l-1} + \dots + a_{k_l}^{(l)}(w)$ where $a_1^{(l)}, \dots, a_{k_l}^{(l)}$ are holomorphic near 0 and vanish at 0. Moreover, since the intersection $X_1 \cap X_2$ is transverse, P_1 and P_2 are relatively prime. Thus there exists two polynomials α_1 and α_2 with holomorphic coefficients in w and a function β of w not identically zero such that

$$\alpha_1(z, w)f_1(z, w) + \alpha_2(z, w)f_2(z, w) = \beta(w).$$

Multiplying this equality by φ_1 we get

$$f_2(\alpha_1\varphi_2 + \alpha_2\varphi_1) = \beta\varphi_1.$$

We now prove that β divides the function $\psi := \alpha_1\varphi_2 + \alpha_2\varphi_1$.

Since β is not identically zero, there exists $k \in \mathbb{N}$ such that $\beta(w) = w^k \gamma(w)$ where $\gamma(0) \neq 0$.

For all $j \in \mathbb{N}$ we have

$$(10) \quad f_2(z, w) \frac{\partial^j \psi}{\partial \bar{w}^j}(z, w) = \beta(w) \frac{\partial \varphi_1}{\partial \bar{w}^j}(z, w)$$

and for $w = 0$ and all z we thus get $\frac{\partial^j \psi}{\partial \bar{w}^j}(z, 0) = 0$.

By induction we then deduce from (10) that $\frac{\partial^{i+j} \psi}{\partial w^i \partial \bar{w}^j}(z, 0) = 0$ for all $i \in \{0, \dots, k-1\}$ and all $j \in \mathbb{N}$. For any integer $n \geq k$ we therefore can write for all z and all w

$$\frac{\psi(z, w)}{w^k} = \sum_{\substack{k \leq i+j \leq n \\ i \geq k}} w^{i-k} \bar{w}^j \frac{\partial^{i+j} \psi}{\partial w^i \partial \bar{w}^j}(z, 0) + \sum_{i+j=n+1} w^{i-k} \bar{w}^j \int_0^1 \frac{\partial^{n+1} \psi}{\partial w^i \partial \bar{w}^j}(z, tw) dt.$$

Now, it is easy to check by induction that the function $w \mapsto \frac{\bar{w}^{i+j}}{w^i}$ is of class C^{j-1} for all positive integer j and all non negative integer i . This implies that $\frac{\psi(z, w)}{w^k}$ is of class C^n for all positive integer n and therefore $\frac{\varphi_1}{f_2} = \frac{\psi}{\beta}$ is of class C^∞ . \square

5. PROOF OF THE MAIN RESULT

In order to prove Theorem 1.1, for any k and l in $\{1, 2\}$ and any $q \in [1, +\infty]$, we have to prove that if h is a smooth function such that, for all non-negative integers α and β , $\left| \frac{\partial^{\alpha+\beta} \tilde{h}}{\partial \bar{\eta}^\alpha \partial \bar{v}^\beta} \right| |\rho|^{\alpha+\frac{\beta}{2}}$ belongs to $L^q(D)$, then the function

$$z \mapsto \langle T_l, \bar{\partial} h \wedge b_k(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle$$

belongs to $L^q(D)$ if $q < \infty$ and to $BMO(D)$ if $q = +\infty$.

As usually, the main difficulty occurs when z is near bD and when we integrate for ζ near z . Moreover, the only interesting case here is when, in addition, z is near a point $\zeta_0 \in bD \cap X_1 \cap X_2$ and we only consider that case.

We use the same notation as in section 3 and assume that z belongs to the neighborhood \mathcal{U}_0 of a point $\zeta_0 \in bD \cap X_1 \cap X_2$ which was used during the construction of the currents. Moreover, we assume that the *Koranyi* basis at ζ_0 is the canonical basis of \mathbb{C}^2 and that ζ_0 is the origin of \mathbb{C}^2 . We will need upper bound of $\frac{P_l^{(j,k)}}{f_l} \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}$ in order to estimate $\frac{P_l^{(j,k)}}{f_l} b_m$ and the derivatives of $\chi_l^{(j,k)}$. We begin with the following lemma :

Lemma 5.1. *For all $j \in \mathbb{N}$, all $k \in \{1, \dots, n_j\}$, all α and β in \mathbb{N} , $l = 1, 2$, all ζ in $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ and all $\tilde{\zeta} \in \mathbb{C}^2$ such that $|\tilde{\zeta}_1^*| < 2\kappa|\rho(z_{j,k})|$ and $|\tilde{\zeta}_2^*| < (4\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, we have uniformly with respect to j, k, l, ζ and $\tilde{\zeta}$*

$$\left| \frac{P_l^{(j,k)}(\zeta)}{f_l(\zeta)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\frac{f_l(\tilde{\zeta})}{P_l^{(j,k)}(\tilde{\zeta})} \right) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}.$$

Proof: We denote by $(\zeta_{0,1}^*, \zeta_{0,2}^*)$ the coordinates of ζ_0 in the *Koranyi* coordinates at $z_{j,k}$. The definition of $P_l^{(j,k)}$ forces us to distinguish three cases :

First case : If $|\zeta_{0,1}^*| < 2\kappa|\rho(z_{j,k})|$ and $|\zeta_{0,2}^*| < (6\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, then $\delta(z_{j,k}, \zeta_0) \leq 6\kappa|\rho(z_{j,k})|$ and thus for all $\tilde{\zeta} \in \mathcal{P}_{6\kappa|\rho(z_{j,k})|}(z_{j,k})$, $\delta(\tilde{\zeta}, \zeta_0) \lesssim |\rho(z_{j,k})|$.

For all $\varepsilon > 0$ and all $\tilde{\zeta} \in \mathcal{P}_\varepsilon(\zeta_0)$, it is easy to see that $|f_l(\tilde{\zeta})| \lesssim \varepsilon^{\frac{p_l}{2}}$. Therefore, Cauchy's inequalities give

$$\left| \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\tilde{\zeta}) \right| \lesssim |\rho(z_{j,k})|^{\frac{p_l}{2} - \alpha - \frac{\beta}{2}}$$

for all $\zeta \in \mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$. Moreover, since $|\zeta_{0,1}^*| < 2\kappa|\rho(z_{j,k})|$, on the one hand $P_l^{(j,k)} = 1$, and on the other hand $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l = \emptyset$ (see Proposition 3.1) which implies that $|f_l(\zeta)| \gtrsim |\rho(z_{j,k})|^{\frac{p_l}{2}}$ for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$. Therefore $\left| \frac{P_l^{(j,k)}(\zeta)}{f_l(\zeta)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\frac{f_l(\tilde{\zeta})}{P_l^{(j,k)}(\tilde{\zeta})} \right) \right| \lesssim |\rho(z_{j,k})|^{-\alpha - \frac{\beta}{2}}$.

Second case : If $|\zeta_{0,1}^*| < 2\kappa|\rho(z_{j,k})|$ and $|\zeta_{0,2}^*| \geq (6\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, we set $a(z_{j,k}) = \frac{\partial \rho}{\partial \zeta_1}(z_{j,k})$, $b(z_{j,k}) = \frac{\partial \rho}{\partial \zeta_2}(z_{j,k})$ and

$$P(z_{j,k}) = \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} \begin{pmatrix} a(z_{j,k}) & b(z_{j,k}) \\ -\bar{b}(z_{j,k}) & \bar{a}(z_{j,k}) \end{pmatrix}.$$

Then we have $\zeta^* = P(z_{j,k})(\zeta - z_{j,k})$. Moreover $b(z_{j,k})$ tends to 0 when $z_{j,k}$ goes to ζ_0 , that is if \mathcal{U}_0 is sufficiently small.

For $\tilde{\zeta} \in \mathcal{P}_{5\kappa|\rho(z_{j,k})|}(z_{j,k})$, if \mathcal{U}_0 is sufficiently small

$$\begin{aligned} |\tilde{\zeta}_2| &\geq \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} (|a(z_{j,k})||\zeta_{0,2}^*| - |b(z_{j,k})||\zeta_{0,1}^*| - |b(z_{j,k})||\tilde{\zeta}_1^*| - |a(z_{j,k})||\tilde{\zeta}_2^*|) \\ &\gtrsim |\zeta_{0,2}^*|. \end{aligned}$$

We also trivially have $|\tilde{\zeta}_2| \lesssim |\zeta_{0,2}^*|$ and so $|\tilde{\zeta}_2| \approx |\zeta_{0,2}^*|$. Analogously we have $|\zeta_2| \approx |\zeta_{0,2}^*|$ for $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$.

On the other hand

$$\begin{aligned} |\tilde{\zeta}_1| &\leq \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} (|a(z_{j,k})|(|\zeta_{0,1}^*| + |\tilde{\zeta}_1^*|) + |b(z_{j,k})|(|\zeta_{0,2}^*| + |\tilde{\zeta}_2^*|)) \\ &\leq 2\kappa|\rho(z_{j,k})| + |b(z_{j,k})|(|\zeta_{0,2}^*| + |\rho(z_{j,k})|^{\frac{1}{2}}) \\ &\leq c|\zeta_{0,2}^*| \end{aligned}$$

where c does not depend from $z_{j,k}$ nor from $\tilde{\zeta}$ and is arbitrarily small provided \mathcal{U}_0 is small enough. We also have $|\zeta_1| \leq c|\zeta_{0,2}^*|$ for $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$.

Now let $\alpha \in \mathbb{C}$ be such that $f_l(\zeta_1, \alpha) = 0$. Since the intersection $X_l \cap bD$ is transverse, there exists a positive constant C not depending from $\tilde{\zeta}$, α , j nor k such that $|\alpha| \leq C|\tilde{\zeta}_1|$. Therefore if c is small enough, $|\alpha| \leq \frac{1}{2}|\tilde{\zeta}_2|$. This yields

$$\begin{aligned} |f_l(\tilde{\zeta})| &\approx \prod_{\alpha/f_l(\tilde{\zeta}_1, \alpha)=0} |\tilde{\zeta}_2 - \alpha| \\ &\approx |\zeta_{0,2}^*|^{p_l}. \end{aligned}$$

Analogously we have $|f_l(\zeta)| \approx |\zeta_{0,2}^*|^{p_l}$ for $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$. Cauchy's inequalities then give for all $\tilde{\zeta} \in \mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$

$$\left| \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\zeta_{0,2}^*|^{p_l} |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}},$$

and since $P_l^{(j,k)} = 1$ when $|\zeta_{0,1}^*| \leq 2\kappa|\rho(z_{j,k})|$, we are done in this case.

Third case : If $|\zeta_{0,1}^*| > 2\kappa|\rho(z_{j,k})|$, there exists a family of parametrization $\alpha_{l,i}^{(j,k)}$, $i = 1, \dots, p_l$, given by Proposition 3.1 such that $\left| \frac{\partial^n \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^{*n}}(\zeta_1^*) \right| \lesssim |\rho(z_{j,k})|^{1-n}$ for all $\zeta_1^* \in \Delta_0(2\kappa|\rho(z_{j,k})|)$. Moreover in this case, we actually seek an upper bound for

$$\frac{1}{\prod_{i \notin I_l^{(j,k)}} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*))} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\prod_{i \in I_l^{(j,k)}} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)) \right).$$

We fix i in $\{1, \dots, p_l\} \setminus I_l^{(j,k)}$ and $\tilde{\zeta}$ such that $|\tilde{\zeta}_1^*| < 2\kappa|\rho(z_{j,k})|$ and $|\tilde{\zeta}_2^*| < (4\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$. If $|\alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)| \leq (6\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, we have $|\tilde{\zeta}_2^* - \alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)| \lesssim (\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$. On the other hand, by definition of $I_l^{(j,k)}$, for all $\zeta_1^* \in \Delta_0(2\kappa|\rho(z_{j,k})|)$, we have $|\alpha_{l,i}^{(j,k)}(\zeta_1^*)| \geq (2\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$. Therefore, for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ we have

$$(11) \quad \frac{|\tilde{\zeta}_2^* - \alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)|}{|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)|} \lesssim 1$$

uniformly with respect to $\zeta, \tilde{\zeta}$ and $z_{j,k}$.

If now $|\alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)| \geq (6\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, we have $|\tilde{\zeta}_2^* - \alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)| \lesssim |\alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)|$.

If \mathcal{U}_0 is sufficiently small, Proposition 2.2 then yields $|\alpha_{l,i}^{(j,k)}(\zeta_1^*)| \gtrsim (3\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$ for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ and

$$(12) \quad \frac{|\tilde{\zeta}_2^* - \alpha_{l,i}^{(j,k)}(\tilde{\zeta}_1^*)|}{|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)|} \lesssim 1$$

uniformly with respect to $\zeta, \tilde{\zeta}$ and $z_{j,k}$.

From proposition 2.2 we also have

$$(13) \quad \frac{1}{|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)|} \lesssim |\rho(z_{j,k})|^{-\frac{1}{2}}$$

for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ and $\left| \frac{\partial^\alpha \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^{*\alpha}}(\tilde{\zeta}_1^*) \right| \lesssim |\rho(z_{j,k})|^{1-\alpha}$ for all $\tilde{\zeta}_1^* \in \Delta_0(2\kappa|\rho(z_{j,k})|)$ so

$$(14) \quad \frac{1}{|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)|} \left| \frac{\partial^\alpha \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^{*\alpha}}(\tilde{\zeta}_1^*) \right| \lesssim |\rho(z_{j,k})|^{-\alpha}.$$

Now the inequalities (11), (12), (13) and (14) yield the lemma. \square

Lemma 5.1 gives us an upper bound for the derivatives of $\chi_l^{(j,k)}$:

Corollary 5.2. *For all $j \in \mathbb{N}$, all $k \in \{1, \dots, n_j\}$, all α and β in \mathbb{N} , $l = 1, 2$ and all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, we have uniformly with respect to j, k, l and ζ*

$$\left| \frac{\partial^{\alpha+\beta} \chi_l^{(j,k)}}{\partial \bar{\zeta}_1^{\alpha} \partial \bar{\zeta}_2^{\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}.$$

Proof: Since by construction $\left| \frac{\partial^{\alpha+\beta} \tilde{\chi}_{j,k}}{\partial \bar{\zeta}_1^{\alpha} \partial \bar{\zeta}_2^{\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}$, we only have to consider

$$\frac{\partial^{\alpha+\beta}}{\partial \bar{\zeta}_1^{\alpha} \partial \bar{\zeta}_2^{\beta}} \chi \left(\frac{f_1(\zeta)}{P_1^{(j,k)}(\zeta)} |\rho(z_{j,k})|^{i_1^{(j,k)}}, \frac{f_2(\zeta)}{P_2^{(j,k)}(\zeta)} |\rho(z_{j,k})|^{i_2^{(j,k)}} \right).$$

The derivative $\frac{\partial^{\gamma+\delta} \chi}{\partial z_1^{\gamma} \partial z_2^{\delta}}(z_1, z_2)$ is bounded up to a uniform multiplicative constant by $\frac{1}{|z_1|^{\gamma} |z_2|^{\delta}}$ when $\frac{1}{3}|z_2| < |z_1| < \frac{2}{3}|z_2|$ and is zero otherwise.

Therefore, we can estimate $\left| \frac{\partial^{\alpha+\beta} \chi_l^{(j,k)}}{\partial \bar{\zeta}_1^{\alpha} \partial \bar{\zeta}_2^{\beta}} \right|$ by a sum of products of $\left| \frac{P_l^{(j,k)}}{f_l} \frac{\partial^{\gamma+\delta}}{\partial \bar{\zeta}_1^{\gamma} \partial \bar{\zeta}_2^{\delta}} \left(\frac{f_l}{P_l^{(j,k)}} \right) \right|$ where the sum of the γ 's equals α and the sum of the δ 's equals β . Lemma 5.1 then gives the wanted estimates. \square

Corollary 5.3. *For any smooth function h , we can write*

$$\frac{\partial^{i_l^{(j,k)}}}{\partial \bar{\zeta}_2^{i_l^{(j,k)}}} \left(\chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) \wedge P^{N,1}(\zeta, z) \right) = \psi_1^{(j,k,l)}(\zeta, z) d\zeta_1^* + \psi_2^{(j,k,l)}(\zeta, z) d\zeta_2^*$$

with $\psi_1^{(j,k,l)}$ and $\psi_2^{(j,k,l)}$ two $(0,2)$ -forms supported in $\mathcal{U}_l^{(j,k)}$ satisfying for ∇_z a differential operator of order 1 acting on z , uniformly with respect to j, k, z and $\zeta \in \mathcal{U}_l^{(j,k)}$:

$$\begin{aligned} \left| \psi_1^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-i_l^{(j,k)}-\frac{5}{2}} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \\ \left| \psi_2^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-i_l^{(j,k)}-2} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \\ \left| \nabla_z \psi_1^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-i_l^{(j,k)}-\frac{7}{2}} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \\ \left| \nabla_z \psi_2^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-i_l^{(j,k)}-3} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \end{aligned}$$

where $\tilde{h}(\zeta) = \max_{n \in \{0, \dots, i_l^{(j,k)}\}} \left(\left| \frac{\partial^{n+1} h}{\partial \bar{\zeta}_2^{n+1}}(\zeta) |\rho(\zeta)|^{\frac{n+1}{2}} \right|, \left| \frac{\partial^{n+1} h}{\partial \bar{\zeta}_1^n \partial \bar{\zeta}_2^n}(\zeta) |\rho(\zeta)|^{\frac{n}{2}+1} \right| \right).$

Proof: Proposition 2.4 implies that $\frac{\partial^n}{\partial \bar{\zeta}_2^n} P^{N,1}(\zeta, z) = \sum_{p,q=1,2} \tilde{\psi}_{p,q}^{(n,N)}(\zeta, z) d\zeta_p^* \wedge d\bar{\zeta}_q^*$ where

$$|\tilde{\psi}_{p,q}^{n,N}(\zeta, z)| \lesssim \left(\frac{|\rho(\zeta)|}{|\rho(\zeta)| + |\rho(z)| + \delta(\zeta, z)} \right)^N |\rho(\zeta)|^{-\frac{1}{p}-\frac{1}{q}-\frac{n}{2}}.$$

From proposition 2.2, if κ is small enough, we have for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, $\frac{1}{2}|\rho(z_{j,k})| \leq |\rho(\zeta)|$ and thus, provided κ is small enough :

$$\begin{aligned} |\rho(\zeta)| + \delta(\zeta, z) &\geq \frac{1}{2}|\rho(z_{j,k})| + \frac{1}{c_1}\delta(z, z_{j,k}) - \delta(z_{j,k}, \zeta) \\ &\gtrsim |\rho(z_{j,k})| + \delta(z, z_{j,k}) \end{aligned}$$

and so $|\tilde{\psi}_{p,q}^{n,N}(\zeta, z)| \lesssim \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N |\rho(z_{j,k})|^{-\frac{1}{p} - \frac{1}{q} - \frac{n}{2}}$. This inequality and Corollary 5.2 now yield the two first estimates. The two others can be shown in the same way.

□

In order to estimate $\frac{P_l^{(j,k)}}{f_l} b_k$, we need the following lemma :

Lemma 5.4. *For all $j \in \mathbb{N}$, all $k \in \{1, \dots, n_j\}$, all α and β in \mathbb{N} , $l = 1, 2$ and all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ we have uniformly with respect to j, k, l and ζ*

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\prod_{i \in I_l^{(j,k)}} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)) \right) \right| \lesssim |\rho(z_{j,k})|^{i_l^{(j,k)} - \alpha - \frac{\beta}{2}}.$$

Proof: For $i \in I_l^{(j,k)}$, there exists $z_1^* \in \Delta_0(\kappa|\rho(z_{j,k})|)$ such that $|\alpha_{l,i}^{(j,k)}(z_1^*)| < 2\kappa|\rho(z_{j,k})|^{\frac{1}{2}}$. Since $\left| \frac{\partial \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^*}(\zeta_1^*) \right|$ is uniformly bounded on $\Delta_0(2\kappa|\rho(z_{j,k})|)$, for all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$, we have $\prod_{i \in I_l^{(j,k)}} |\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \lesssim |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2}}$. Cauchy's inequalities then give the results.

□

As a direct corollary of Lemma 5.1 and 5.4 we get

Corollary 5.5. *For all $j \in \mathbb{N}$, all $k \in \{1, \dots, n_j\}$, all α and β in \mathbb{N} , $l = 1, 2$ and all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ we have uniformly with respect to j, k, l and ζ*

$$\left| \frac{P_l^{(j,k)}(\zeta)}{f_l(\zeta)} \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{i_l^{(j,k)} - \alpha - \frac{\beta}{2}}.$$

In the proof of the following corollary appears the technical reason why we have to introduce the open sets $\mathcal{U}_1^{j,k}$ and $\mathcal{U}_2^{j,k}$.

Corollary 5.6. *For $l, m \in \{1, 2\}$, we can write $\frac{P_l^{(j,k)}}{f_l} b_m = \varphi_1^{(j,k,l,m)} d\zeta_1^* + \varphi_2^{(j,k,l,m)} d\zeta_2^*$ with $\varphi_1^{(j,k,l,m)}$ and $\varphi_2^{(j,k,l,m)}$ satisfying for all $\zeta \in \mathcal{U}_l^{(j,k)}$ and all differential operator ∇_z of order 1 acting on z ,*

$$\begin{aligned} \left| \varphi_1^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \\ \left| \varphi_2^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - \frac{1}{2}} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \\ \left| \nabla_z \varphi_1^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - 2} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \\ \left| \nabla_z \varphi_2^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - \frac{3}{2}} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \end{aligned}$$

uniformly with respect to ζ, z, j and k .

Proof: Without restriction we assume $l = 1$ and for $m = 1, 2$, we write $b_m(\zeta, z) = b_{m,1}^*(\zeta, z)d\zeta_1^* + b_{m,2}^*(\zeta, z)d\zeta_2^*$ where $b_{m,n}^* = \int_0^1 \frac{\partial f_m}{\partial \zeta_n^*}(\zeta + t(z - \zeta))dt$. So

$$b_{m,n}^*(\zeta, z) = \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} \frac{1}{\alpha + \beta + 1} \frac{\partial^{\alpha + \beta + 1} f_m}{\partial \zeta_n^* \partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) (z_1^* - \zeta_1^*)^\alpha (z_2^* - \zeta_2^*)^\beta + o(|z - \zeta|^{\max(p_1, p_2)})$$

and Corollary 5.5 yields for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$:

$$\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{1,1}(\zeta, z) \right| \lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}$$

uniformly with respect to z, ζ, j and k . The proof of the inequality for $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{1,2}(\zeta, z) \right|$

is exactly the same. The one for $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,1}(\zeta, z) \right|$ uses the definition of $\mathcal{U}_1^{(j,k)}$.

On $\mathcal{U}_1^{(j,k)}$, we have $\left| \frac{P_1^{(j,k)}}{f_1} \right| \lesssim \left| \frac{P_2^{(j,k)}}{f_2} \right| |\rho(z_{j,k})|^{i_1^{(j,k)} - i_2^{(j,k)}}$ and again Corollary 5.5 yields

$$\begin{aligned} \left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,1}(\zeta, z) \right| &\lesssim \left| \frac{P_2^{(j,k)}(\zeta)}{f_2(\zeta)} b_{2,1}(\zeta, z) \right| |\rho(z_{j,k})|^{i_1^{(j,k)} - i_2^{(j,k)}} \\ &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{i_l^{(j,k)} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}} \end{aligned}$$

uniformly with respect to z, ζ, j and k . Again, the inequality for $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,2}(\zeta, z) \right|$ can be obtained in the same way. \square

Corollary 5.6 and 5.3 imply for some N' arbitrarily large provided N is large enough, that

$$\begin{aligned} \left| \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} b_m(\zeta, z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial \zeta_2^{*i_l^{(j,k)}}} \left(\chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) P^{N,1}(\zeta, z) \right) \right| \\ \leq |\rho(z_{j,k})|^{-3} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^{N'} \tilde{h}(\zeta) \end{aligned}$$

and for ∇_z a differential of order 1

$$\begin{aligned} \left| \nabla_z \left(\frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} b_m(\zeta, z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial \zeta_2^{*i_l^{(j,k)}}} \left(\chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) P^{N,1}(\zeta, z) \right) \right) \right| \\ \leq |\rho(z)|^{-1} |\rho(z_{j,k})|^{-3} \left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^{N'} \tilde{h}(\zeta) \end{aligned}$$

where $\tilde{h}(\zeta) = \max_{n \in \{0, \dots, i_l^{(j,k)}\}} \left(\left| \frac{\partial^{n+1} h}{\partial \zeta_2^{*n+1}}(\zeta) |\rho(\zeta)|^{\frac{n+1}{2}} \right|, \left| \frac{\partial^{n+1} h}{\partial \zeta_1^* \partial \zeta_2^{*n}}(\zeta) |\rho(\zeta)|^{\frac{n}{2}+1} \right| \right)$. We conclude as in [1] that Theorem 1.1 holds true.

6. LOCAL DIVISION

6.1. Local holomorphic division. In this subsection we will prove Theorem 1.2 and his analogue in the L^q case, the following

Theorem 6.1. *When $n = 2$, let g be a holomorphic function defined on D . Assume that $X_1 \cap X_2$ is a complete intersection and that there exist $\kappa > 0$, a real number $q \geq 1$ and a locally finite covering $(\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j))_{j \in I}$ of D such that for all $j \in I$, there exist two function $\hat{g}_1^{(j)}$ and $\hat{g}_2^{(j)}$, C^∞ -smooth on $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$, such that*

- (a) $g = \hat{g}_1^{(j)} f_1 + \hat{g}_2^{(j)} f_2$ on $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$;
- (b) $c_{l,\alpha,\beta} := \sum_{j \in I} \int_{\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)} \left| \frac{\partial^{\alpha+\beta} \hat{g}_l^{(j)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(z) \right|^q |\rho(z)|^{\alpha+\frac{\beta}{2}} dV(z) < \infty$ for $l = 1$ and $l = 2$ and all integers α and β ;
- (c) for $l = 1$ and $l = 2$, for all non negatives integers $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$, there exist $N \in \mathbb{N}$ and $c > 0$ such that for all j , $\sup_{\mathcal{P}_{\kappa|\rho(z)|}(z)} \left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \hat{g}_l^{(j)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \zeta_1^{*\bar{\alpha}} \partial \zeta_2^{*\bar{\beta}}}(z) \right| |\rho(z)|^N \leq c$.

Then there exist two smooth functions \tilde{g}_1 and \tilde{g}_2 which satisfy (i)-(iii) of Theorem 1.1 with q .

Proof: It suffices to glue together all the $\hat{g}_1^{(j)}$ and $\hat{g}_2^{(j)}$ using a suitable partition of unity. Let $(\chi_j)_{j \in \mathbb{N}}$ be a partition of unity subordinated to $(\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j))_{j \in \mathbb{N}}$ such that for all j and all $\zeta \in \mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$, we have $\left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \chi_j}{\partial z_1^{*\alpha} \partial z_2^{*\beta} \partial z_1^{*\bar{\alpha}} \partial z_2^{*\bar{\beta}}}(\zeta) \right| \lesssim \frac{1}{|\rho(\zeta_j)|^{\alpha+\bar{\alpha}+\frac{\beta+\bar{\beta}}{2}}}$, uniformly with respect to ζ_j and ζ . We set $\tilde{g}_1 = \sum_j \chi_j \hat{g}_1^{(j)}$ and $\tilde{g}_2 = \sum_j \chi_j \hat{g}_2^{(j)}$ and thus we get the two functions defined on D which satisfy (i), (ii) and (iii) by construction. \square

Theorem 1.2 can be proved in exactly the same way than Theorem 6.1, so we omit the proof.

6.2. Divided differences and division. In order to apply Theorem 1.2 and 6.1, we will use divided differences and find numerical conditions on g which ensure the existence of local smooth division formula in L^∞ or in L^q . We define the divided differences using the following settings.

We set

$$\Lambda_{z,v}^{(1)} = \{\lambda \in \mathbb{C}, |\lambda| < \tau(z, v, 3\kappa|\rho(z)|) \text{ and } z + \lambda v \in X_2 \setminus X_1\}$$

The points $z + \lambda v$, $\lambda \in \Lambda_{z,v}^{(1)}$, are the points of $X_2 \setminus X_1$ which belong to $\Delta_{z,v}(\tau(z, v, 3\kappa|\rho(z)|))$, thus they all belong to D as soon as $\kappa < \frac{1}{3}$. We analogously define

$$\Lambda_{z,v}^{(2)} = \{\lambda \in \mathbb{C}, |\lambda| < \tau(z, v, 3\kappa|\rho(z)|) \text{ and } z + \lambda v \in X_1 \setminus X_2\}.$$

For a function h defined on a subset \mathcal{U} of \mathbb{C}^n , $z \in \mathbb{C}^n$, v a unit vector of \mathbb{C}^n and $\lambda \in \mathbb{C}$ such that $z + \lambda v$ belongs to \mathcal{U} , we set $h_{z,v}[\lambda] = h(z + \lambda v)$. If $h_{z,v}[\lambda_1, \dots, \lambda_k]$ is defined, for

$\lambda_1, \dots, \lambda_{k+1} \in \mathbb{C}$ pairwise distinct such that $z + \lambda_i v$ belongs to \mathcal{U} for all i , we set

$$h_{z,v}[\lambda_1, \dots, \lambda_{k+1}] := \frac{h_{z,v}[\lambda_1, \dots, \lambda_k] - h_{z,v}[\lambda_2, \dots, \lambda_{k+1}]}{\lambda_1 - \lambda_{k+1}}.$$

Now, for $z \in X_2 \setminus X_1$ (resp. $z \in X_1 \setminus X_2$) let us define $g^{(2)}(z) = \frac{g(z)}{f_2(z)}$ (resp. $g^{(1)}(z) = \frac{g(z)}{f_1(z)}$). For $l = 1$ or $l = 2$, the quantity $g_{z,v}^{(l)}[\lambda_1, \dots, \lambda_k]$ make sense for all $\lambda_1, \dots, \lambda_k \in \Lambda_{z,v}^{(l)}$ pairwise distinct.

We first prove a technical result we will need in this section.

Lemma 6.2. *Let α and β be two functions defined on a subset \mathcal{U} of \mathbb{C} . Then, for all z_1, \dots, z_n pairwise distinct points of \mathcal{U} we have*

$$(\alpha \cdot \beta)[z_1, \dots, z_n] = \sum_{k=1}^n \alpha[z_1, \dots, z_k] \cdot \beta[z_k, \dots, z_n].$$

Proof: We prove the lemma by induction on n , the case $n = 1$ being trivial. We assume the lemma proved for n points, $n \geq 1$. Let z_1, \dots, z_{n+1} be $n + 1$ points of \mathcal{U} . Then

$$\begin{aligned} & (\alpha \cdot \beta)[z_1, \dots, z_{n+1}] \\ &= \frac{(\alpha \cdot \beta)[z_1, z_3, \dots, z_{n+1}] - (\alpha \cdot \beta)[z_2, \dots, z_{n+1}]}{z_1 - z_2} \\ &= \frac{1}{z_1 - z_2} \left(\sum_{k=3}^{n+1} \alpha[z_1, z_3, \dots, z_k] \beta[z_k, \dots, z_{n+1}] + \alpha[z_1] \beta[z_3, \dots, z_{n+1}] \right) \\ &\quad - \frac{1}{z_1 - z_2} \sum_{k=2}^{n+1} \alpha[z_2, \dots, z_k] \beta[z_k, \dots, z_{n+1}] \\ &= \sum_{k=3}^{n+1} \frac{\alpha[z_1, z_3, \dots, z_k] - \alpha[z_2, \dots, z_k]}{z_1 - z_2} \beta[z_k, \dots, z_{n+1}] + \\ &\quad \frac{\alpha[z_1] - \alpha[z_2]}{z_1 - z_2} \beta[z_3, \dots, z_{n+1}] + \alpha[z_1] \frac{\beta[z_3, \dots, z_{n+1}] - \beta[z_2, \dots, z_{n+1}]}{z_1 - z_2}. \quad \square \end{aligned}$$

6.2.1. The $L^\infty - BMO$ -case. In this subsection, we establish the necessary conditions in \mathbb{C}^n and the sufficient conditions \mathbb{C}^2 for a function g to be written as $g = g_1 f_1 + g_2 f_2$, g_1 and g_2 smooth functions satisfying the hypothesis of Theorem 1.1.

For $l = 1$ and $l = 2$ let us define the numbers

$$c_\infty^{(l)}(g) = \sup \left(|g_{z,v}^{(l)}[\lambda_1, \dots, \lambda_k]| \tau(z, v, |\rho(z)|)^{k-1} \right)$$

where the supremum is taken over all $z \in D$, all $v \in \mathbb{C}^n$ with $|v| = 1$ and all $\lambda_1, \dots, \lambda_k \in \Lambda_{z,v}^{(l)}$ pairwise distinct.

We have the following necessary conditions in \mathbb{C}^n , $n \geq 2$.

Theorem 6.3. *In \mathbb{C}^n , $n \geq 2$, let g be a holomorphic on D and let g_1, g_2 be two bounded holomorphic functions on D such that $g = g_1 f_1 + g_2 f_2$. Then*

$$\left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^\infty(D)} \lesssim \max(\|g_1\|_{L^\infty(D)}, \|g_2\|_{L^\infty(D)})$$

and for $l = 1, 2$:

$$c_\infty^{(l)}(g) \lesssim \sup_{b\Delta_{z,v}(4\kappa\tau(z,v,|\rho(z)|))} |g_l|.$$

Proof : The first point is trivial and we only prove the second one for $l = 1$. Let $\lambda_1, \dots, \lambda_k$ be k pairwise distinct elements of $\Lambda_{z,v}^{(1)}$. For all i we have $g_{z,v}^{(1)}[\lambda_i] = g_1(z + \lambda_i v)$ because $f_2(z + \lambda_i v) = 0$. Therefore, $g_{z,v}^{(1)}[\lambda_1, \dots, \lambda_k] = (g_1)_{z,v}[\lambda_1, \dots, \lambda_k]$. As in [1], it then follows from Cauchy's formula that

$$\begin{aligned} |g_{z,v}^{(1)}[\lambda_1, \dots, \lambda_k]| &\lesssim \left| \frac{1}{2i\pi} \int_{|\lambda|=\tau(z,v,4\kappa|\rho(z)|)} \frac{g_1(z + \lambda v)}{\prod_{i=1}^k (\xi - \lambda_i)} d\xi \right| \\ &\lesssim \tau(z, v, |\rho(z)|)^{-k+1} \sup_{b\Delta_{z,v}(4\kappa\tau(z,v,|\rho(z)|))} |g_1|. \quad \square \end{aligned}$$

Now we prove that these conditions are sufficient in \mathbb{C}^2 in order to get a *BMO* division.

Theorem 6.4. *In \mathbb{C}^2 , let g be a holomorphic function on D which belong to the ideal generated by f_1 and f_2 and such that*

- (i) $c(g) = \sup_{z \in D} \frac{|g(z)|}{\max(|f_1(z)|, |f_2(z)|)} < \infty$,
- (ii) $c_\infty^{(1)}(g)$ and $c_\infty^{(2)}(g)$ are finished.

Then for all $z \in D$, there exist two holomorphic functions g_1 and g_2 which belong to $BMO(D)$ and such that $g_1 f_1 + g_2 f_2 = g$.

Proof : It suffices to construct for all z near bD two smooth functions \hat{g}_1 and \hat{g}_2 on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ which satisfy (a) and (b) of Theorem 1.2.

Let ζ_0 be a point in bD . If $f_1(\zeta_0) \neq 0$ then f_1 does not vanish on a neighborhood \mathcal{U}_0 of ζ_0 . Then we can define $\hat{g}_1 = \frac{g}{f_1}$, $\hat{g}_2 = 0$ which obviously satisfy (a) and (b) for all $z \in D$ close to ζ_0 . We proceed analogously if $f_2(\zeta_0) \neq 0$.

If ζ_0 belongs to $X_1 \cap X_2 \cap bD$, since the intersection $X_1 \cap X_2$ is complete, without restriction we can choose a neighborhood \mathcal{U}_0 of ζ_0 such that $X_1 \cap X_2 \cap \mathcal{U}_0 = \{\zeta_0\}$. Then we fix some point z in \mathcal{U}_0 and we construct \hat{g}_1 and \hat{g}_2 on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ which satisfy (a) and (b) of Theorem 1.2. We denote by p_1 and p_2 the order of ζ_0 as zero of f_1 and f_2 respectively. We also denote by $(\zeta_{0,1}^*, \zeta_{0,2}^*)$ the coordinates of ζ_0^* in the Koranyi coordinates at z . If $|\zeta_{0,1}^*| < 2\kappa|\rho(z)|$, then for $l = 1$ and $l = 2$ we set $i_l = 0$, $P_l(\zeta) = 1$ and $Q_l(\zeta) = f_l(\zeta)$. Otherwise, we use the parametrization $\alpha_{1,i}$, $i \in \{1, \dots, p_1\}$, of X_1 and $\alpha_{2,i}$, $i \in \{1, \dots, p_2\}$, of X_2 given by Proposition 2.2. We denote by I_l the set $I_l = \{i, \exists z_1^* \in \Delta_0(\kappa|\rho(z)|) \text{ such that } |\alpha_{l,i}(z_1^*)| \leq (2\kappa|\rho(z)|)^{\frac{1}{2}}\}$, $i_l = \#I_l$, $P_l(\zeta) = \prod_{i \in I_l} (\zeta_2^* - \alpha_{l,i}(\zeta_1^*))$ and $Q_l(\zeta) = \frac{f_l}{P_l}$.

If $i_1 = 0$ we set $\tilde{g}_2 = 0$. Otherwise, without restriction we assume that $I_1 = \{1, \dots, i_1\}$ and for $k \leq i_1$ and ζ_1^* such that $f_2(z + \zeta_1^* \eta_z + \alpha_{1,i}(\zeta_1^*) v_z) \neq 0$, we introduce $g^{(2)} = \frac{g}{f_2}$ and

$$(15) \quad \tilde{g}_{1,\dots,k}^{(2)}(\zeta_1^*) := \left(\frac{g}{P_2} \right)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)].$$

Since $X_1 \cap X_2 \cap \mathcal{U}_0 = \{\zeta_0\}$, $\check{g}_{1,\dots,k}^{(2)}$ is defined on $\Delta_0(2\kappa|\rho(z)|)$ and we have by Lemma 6.2

$$\begin{aligned} & \check{g}_{1,\dots,k}^{(2)}(\zeta_1^*) \\ &= \left(\frac{g}{P_2} \right)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)] \\ &= \left(\frac{g}{f_2} Q_2 \right)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)] \\ &= \sum_{j=1}^k g_{z+\zeta_1^* \eta_z, v_z}^{(2)} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,j}(\zeta_1^*)] (Q_2)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,j}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)]. \end{aligned}$$

Now from [17] we have

$$|(Q_2)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,j}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)]| \lesssim |\rho(z)|^{\frac{j-k}{2}} \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^* \eta_z + \xi v_z)|,$$

which, with the assumption $c_\infty^{(2)}(g) < \infty$, gives for all $\zeta_1^* \in \Delta_0(2\kappa|\rho(z)|)$:

$$(16) \quad |\check{g}_{1,\dots,k}^{(2)}(\zeta_1^*)| \lesssim c_\infty^{(2)}(g) |\rho(z)|^{\frac{1-k}{2}} \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^* \eta_z + \xi v_z)|.$$

Now we set

$$\tilde{g}_2(\zeta) = \sum_{k=1}^{i_2} \check{g}_{1,\dots,k}^{(2)}(\zeta_1^*) \prod_{i=1}^{k-1} (\zeta_2^* - \alpha_{1,i}(\zeta_1^*)).$$

and we define \tilde{g}_1 analogously. For ζ_1^* fixed, $\tilde{g}_2(\zeta_1^*, \cdot)$ is the polynomial which interpolates $\frac{g(\zeta_1^*, \cdot)}{P_2(\zeta_1^*, \cdot)}$ at the points $\zeta_2^* = \alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,i_1}(\zeta_1^*)$.

Since $|\alpha_{1,i}(\zeta_1^*)| \lesssim |\rho(z)|^{\frac{1}{2}}$ for all $i \in I_1$ and all $\zeta_1^* \in \Delta_0(2\kappa|\rho(z)|)$, (16) yields for all $\zeta \in \mathcal{P}_{2\kappa|\rho(z)|}(z)$:

$$(17) \quad |\tilde{g}_2(\zeta)| \lesssim c_\infty^{(2)}(g) \sup_{\substack{|\xi_2| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}} \\ |\xi_1| \leq 2\kappa|\rho(z)|}} |Q_2(z + \xi_1 \eta_z + \xi_2 v_z)|.$$

Then Cauchy's inequalities gives for all $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$ and all α and β

$$(18) \quad \left| \frac{\partial^{\alpha+\beta} \tilde{g}_2}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim c_\infty^{(2)}(g) |\rho(z)|^{-\alpha-\frac{\beta}{2}} \sup_{\substack{|\xi_2| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}} \\ |\xi_1| \leq 2\kappa|\rho(z)|}} |Q_2(z + \xi_1 \eta_z + \xi_2 v_z)|.$$

Now let g_1 and g_2 be holomorphic functions on D such that $g = f_1 g_1 + f_2 g_2$. Then $\tilde{g}_2(\zeta_1^*, \cdot)$ interpolates $g_2(\zeta_1^*, \cdot) Q_2(\zeta_1^*, \cdot)$ at the points $\alpha_{1,i}(\zeta_1^*)$ for all $i \in I_2$ because for such an i we have by definition

$$\begin{aligned} \tilde{g}_2(\zeta_1^*, \alpha_{1,i}(\zeta_1^*)) &= \frac{g(\zeta_1^*, \alpha_{1,i}(\zeta_1^*))}{P_2(\zeta_1^*, \alpha_{1,i}(\zeta_1^*))} \\ &= g_2(\zeta_1^*, \alpha_{1,i}(\zeta_1^*)) \cdot Q_2(\zeta_1^*, \alpha_{1,i}(\zeta_1^*)) \end{aligned}$$

Therefore we can write

$$(19) \quad g_2(\zeta) = \frac{1}{Q_2(\zeta)} (\tilde{g}_2(\zeta) + P_1(\zeta) \cdot e_1(\zeta))$$

where e_1 is the interpolation error which is given by

$$(20) \quad e_1(\zeta) = \frac{1}{2i\pi} \int_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} \frac{g_2(\zeta_1^*, \xi) Q_2(\zeta_1^*, \xi)}{P_1(\zeta_1^*, \xi) \cdot (\xi - \zeta_2^*)} d\xi.$$

We have an analogous expression for g_1 . We point out that (19) and its analogous for g_1 also holds if $i_1 = 0$ or $i_2 = 0$.

This yields

$$(21) \quad \begin{aligned} g(\zeta) &= f_1(\zeta)g_1(\zeta) + f_2(\zeta)g_2(\zeta) \\ &= P_1(\zeta)\tilde{g}_1(\zeta) + P_2(\zeta)\tilde{g}_2(\zeta) + P_1(\zeta)P_2(\zeta)e(\zeta) \end{aligned}$$

where

$$\begin{aligned} e(\zeta) &= e_1(\zeta) + e_2(\zeta) \\ &= \frac{1}{2i\pi} \int_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} \frac{g(\zeta_1^*, \xi)}{P_1(\zeta_1^*, \xi) \cdot P_2(\zeta_1^*, \xi) \cdot (\xi - \zeta_2^*)} d\xi. \end{aligned}$$

Searching for \hat{g}_1 and \hat{g}_2 such that $g = \hat{g}_1 f_1 + \hat{g}_2 f_2$ in $\mathcal{P}_{\kappa|\rho(z)|}(z)$, since there is a factor $P_1 P_2$ in front e in (21), we can put $P_2 e$ either in \hat{g}_1 with \tilde{g}_1 or we can put $P_1 e$ in \hat{g}_2 with \tilde{g}_2 . But in order to have a good upper bound, we have to cut it in to two pieces in a suitable way. This will be done analogously to the construction of the currents. Let

$$\begin{aligned} \mathcal{U}_1 &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z), \left| \frac{f_1(\zeta)\rho(z)^{i_1}}{P_1(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)\rho(z)^{i_2}}{P_2(\zeta)} \right| \right\}, \\ \mathcal{U}_2 &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z), \frac{2}{3} \left| \frac{f_2(\zeta)\rho(z)^{i_2}}{P_2(\zeta)} \right| > \left| \frac{f_1(\zeta)\rho(z)^{i_1}}{P_1(\zeta)} \right| \right\}. \end{aligned}$$

Let also χ be a smooth function on $\mathbb{C}^2 \setminus \{0\}$ such that $\chi(z_1, z_2) = 1$ if $|z_1| > \frac{2}{3}|z_2|$ and $\chi(z_1, z_2) = 0$ if $|z_1| < \frac{1}{3}|z_2|$. We set $\chi_1(\zeta) = \chi\left(\frac{f_1(\zeta)\rho(z)^{i_1}}{P_1(\zeta)}, \frac{f_2(\zeta)\rho(z)^{i_2}}{P_2(\zeta)}\right)$, $\chi_2(\zeta) = 1 - \chi_1(\zeta)$ and

$$\begin{aligned} \hat{g}_1(\zeta) &= \frac{1}{Q_1(\zeta)} (\tilde{g}_1(\zeta) + \chi_1(\zeta)P_2(\zeta)e(\zeta)), \\ \hat{g}_2(\zeta) &= \frac{1}{Q_2(\zeta)} (\tilde{g}_2(\zeta) + \chi_2(\zeta)P_1(\zeta)e(\zeta)). \end{aligned}$$

Since $Q_2 = \frac{f_2}{P_2}$, Lemma 5.1 and (18) give for all $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$:

$$(22) \quad \left| \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\frac{1}{Q_2(\zeta)} \tilde{g}_2(\zeta) \right) \right| \lesssim c_\infty^{(2)}(g) |\rho(z)|^{-\alpha-\frac{\beta}{2}}$$

From assumption (i), We get for all $\zeta_1^* \in \Delta_0(2\kappa|\rho(z)|)$ and all ξ such that $|\xi| = (4\kappa|\rho(z)|)^{\frac{1}{2}}$:

$$|g(\zeta_1^*, \xi)| \leq c(g) \left(\sup_{\substack{|\xi_1^*| \leq 2\kappa|\rho(z)| \\ |\xi_2^*| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}}}} |f_1(\xi)| + \sup_{\substack{|\xi_1^*| \leq 2\kappa|\rho(z)| \\ |\xi_2^*| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}}}} |f_2(\xi)| \right).$$

And so Cauchy's inequalities yield for all integer α and all $\zeta_1^* \in \Delta_0(\kappa|\rho(z)|)$ and all ξ such that $|\xi| = (4\kappa|\rho(z)|)^{\frac{1}{2}}$

$$\left| \frac{\partial^\alpha g}{\partial \zeta_1^{*\alpha}}(\zeta_1^*, \xi) \right| \lesssim c(g)|\rho(z)|^{-\alpha} \left(|\rho(z)|^{i_1} \sup_{\substack{|\xi_1^*| \leq 2\kappa|\rho(z)| \\ |\xi_2^*| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}}}} |Q_1(\xi)| + |\rho(z)|^{i_2} \sup_{\substack{|\xi_1^*| \leq 2\kappa|\rho(z)| \\ |\xi_2^*| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}}}} |Q_2(\xi)| \right).$$

Therefore, for all $\zeta \in \mathcal{U}_2$ and all non negative integers α and β , Lemma 5.1 gives

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left(\frac{P_1(\zeta)}{Q_2(\zeta)} e(\zeta) \right) \right| \lesssim |\rho(z)|^{-\alpha-\frac{\beta}{2}} c(g).$$

Since for all $\alpha, \beta \in \mathbb{N}$, $\left| \frac{\partial^{\alpha+\beta} \chi_2}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(z) \right| \lesssim |\rho(z_j)|^{-\alpha-\frac{\beta}{2}}$ (see Lemma 5.1), with (22), this yields

$$\left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \hat{g}_2}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \bar{\zeta}_1^{\bar{\alpha}} \partial \bar{\zeta}_2^{\bar{\beta}}}(\zeta) \right| \lesssim |\rho(z)|^{-\alpha-\bar{\alpha}-\frac{\beta+\bar{\beta}}{2}} \left(c(g) + c_\infty^{(2)}(g) \right).$$

The same inequality holds for \hat{g}_1 and we have finally proved that \hat{g}_1 and \hat{g}_2 are smooth functions such that (a) and (b) of Theorem 1.2 hold. \square

6.3. The L^q -case. The assumption under which a function g holomorphic on D can be written as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 being holomorphic on D and belonging to $L^q(D)$ uses a κ -covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j) \right)_{j \in \mathbb{N}}$ in addition to the divided differences.

By transversality of X_1 and bD , and of X_2 and bD , for all j there exists w_j in the complex tangent plane to $bD_{\rho(z_j)}$ such that π_j , the orthogonal projection on the hyperplane orthogonal to w_j passing through z_j , is a covering of X_1 and X_2 . We denote by w_1^*, \dots, w_n^* an orthonormal basis of \mathbb{C}^n such that $w_1^* = \eta_{z_j}$ and $w_n^* = w_j$ and we set $\mathcal{P}'_\varepsilon(z_j) = \{z' = z_j + z_1^* w_1^* + \dots + z_{n-1}^* w_{n-1}^*, |z_1^*| < \varepsilon \text{ and } |z_k^*| < \varepsilon^{\frac{1}{2}}, k = 2, \dots, n-1\}$. We put

$$c_{q,\kappa,(z_j)_{j \in \mathbb{N}}}^{(l)}(g) = \sum_{j=0}^{\infty} \int_{z' \in \mathcal{P}'_{2\kappa|\rho(z_j)|}(z_j)} \sum_{\substack{\lambda_1, \dots, \lambda_k \in \Lambda_{z', w_n^*} \\ \lambda_i \neq \lambda_l \text{ for } i \neq l}} |\rho(z_j)|^{q \frac{k-1}{2} + 1} \left| g_{z', w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right| dV_{n-1}(z')$$

where dV_{n-1} is the Lebesgue measure in \mathbb{C}^{n-1} and $g^{(l)} = \frac{g}{f_l}$, $l = 1$ or $l = 2$.

Now we prove the following necessary conditions

Theorem 6.5. *Let g_1 and g_2 belonging to $L^q(D)$ be two holomorphic functions on D and set $g = g_1 f_1 + g_2 f_2$. Then*

- (i) $\frac{g}{\max(|f_1|, |f_2|)}$ belongs to $L^q(D)$ and $\left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^q(D)} \lesssim \max(\|g_1\|_{L^q(D)}, \|g_2\|_{L^q(D)}).$
- (ii) For $l = 1$ or $l = 2$ and any κ -covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j) \right)_j$, we have $c_{q,\kappa,(z_j)_j}^{(l)}(g) \lesssim \|g_l\|_{L^q(D)}^q,$

Proof: The point (i) is trivial and we only prove (ii). As in the proof of Theorem 6.3, for all $j \in \mathbb{N}$, all $z' \in \mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)$ and all $r \in [\frac{7}{2}\kappa|\rho(z_j)|^{\frac{1}{2}}, 4\kappa|\rho(z_j)|^{\frac{1}{2}}]$ we have

$$g_{z', w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] = \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{g_l(z' + \lambda w_n^*)}{\prod_{i=1}^k (\xi - \lambda_i)} d\xi.$$

After integration for $r \in [(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}}, (4\kappa|\rho(z_j)|)^{\frac{1}{2}}]$, Jensen's inequality yields

$$\left| g_{z', w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right|^q \lesssim |\rho(z_j)|^{\frac{1-k}{2}q-1} \int_{|\lambda| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}} |g_l(z' + \lambda w_n^*)|^q dV_1(\lambda).$$

Now we integrate the former inequality for $z' \in \mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)$ and get

$$\int_{z' \in \mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)} \left| g_{z', w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right|^q |\rho(z_j)|^{\frac{k-1}{2}q+1} dV_{n-1} \lesssim \int_{z \in \mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)} |g_l(z)|^q dV_n(z).$$

Since $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j) \right)_{j \in \mathbb{N}}$ is a κ -covering, we deduce from this inequality that $c_{q, \kappa, (z_j)_{j \in \mathbb{N}}}^{(l)}(g) \lesssim \|g_l\|_{L^q(D)}^q$. \square

Theorem 6.6. *Let g be a holomorphic function on D belonging to the ideal generated by f_1 and f_2 and such that $c_{q, \kappa, (z_j)_{j \in \mathbb{N}}}^{(l)}(g)$ is finite and such that $\frac{g}{\max(|f_1|, |f_2|)}$ belongs to $L^q(D)$. Then there exist two holomorphic functions g_1 and g_2 which belong to $L^q(D)$ and such that $g = g_1 f_1 + g_2 f_2$.*

Proof: We aim to apply Theorem 6.1. For all j in \mathbb{N} , in order to construct on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ two functions $\hat{g}_1^{(j)}$ and $\hat{g}_2^{(j)}$ which satisfy the assumption of Theorem 6.1, we proceed as in the proof of Theorem 6.4. The main difficulty occurs, as in the proof of Theorem 6.4, when we are near a point ζ_0 which belongs to $X_1 \cap X_2 \cap bD$. We denote by $(\zeta_{0,1}^*, \zeta_{0,2}^*)$ the coordinates of ζ_0 in the Koranyi coordinates at z_j . If $|\zeta_{0,1}^*| < 2\kappa|\rho(z_{j0})|$, we set $i_{1,j} = i_{2,j} = 0$, $I_{1,j} = I_{2,j} = \emptyset$, $P_{1,j} = P_{2,j} = 1$, $Q_{1,j} = f_1$ and $Q_{2,j} = f_2$. Otherwise, we use the parametrization $\alpha_{1,i}^{(j)}$, $i \in \{1, \dots, p_1^{(j)}\}$ of X_1 and $\alpha_{2,i}^{(j)}$, $i \in \{1, \dots, p_2^{(j)}\}$ of X_2 given by Proposition 2.2 and for $l = 1$ and $l = 2$, we still denote by $I_{l,j}$ the set $I_{l,j} = \{i, \exists z_1^* \in \Delta_0(\kappa|\rho(z_j)|) \text{ such that } |\alpha_{l,i}^{(j)}(z_1^*)| \leq 2\kappa|\rho(z_j)|^{\frac{1}{2}}\}$, $i_{l,j} = \#I_{l,j}$, $P_{l,j}(\zeta) = \prod_{i \in I_{l,j}} (\zeta_2^* - \alpha_{l,i}^{(j)}(\zeta_1^*))$ and $Q_{l,j} = \frac{f_l}{P_{l,j}}$. We define $\tilde{g}_1^{(j)}$ and $\tilde{g}_2^{(j)}$ as \tilde{g}_1 and \tilde{g}_2 in the proof of Theorem 6.4. Instead of defining $e_1^{(j)}$ and $e_2^{(j)}$ by integrals over the set $\{|\xi| = (4\kappa|\rho(z_j)|)^{\frac{1}{2}}\}$ as we defined e_1 and e_2 in the proof of Theorem 6.4, here we integrate over $\{(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}\}$ and set

$$e^{(j)}(\zeta) = \frac{1}{2\pi(2 - \sqrt{\frac{7}{2}})(\kappa|\rho(z_j)|)} \int_{\{(\frac{7}{2}\kappa|\rho(z)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z)|)^{\frac{1}{2}}\}} \frac{g(z_1^*, \xi)}{P_{1,j}(z_1^*, \xi)P_{2,j}(z_1^*, \xi)(z_2^* - \xi)} dV(\xi).$$

We therefore have for all j and all $z \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$:

$$g(z) = \tilde{g}_1^{(j)}(z)P_{1,j}(z) + \tilde{g}_2^{(j)}(z)P_{2,j}(z) + P_{1,j}(z)P_{2,j}(z)e^{(j)}(z).$$

We split $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ in two parts as in Theorem 6.4 and set

$$\begin{aligned}\mathcal{U}_1^{(j)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j), \left| \frac{f_1(\zeta)\rho(z_j)^{i_{1,j}}}{P_{1,j}(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)\rho(z_j)^{i_{2,j}}}{P_2(\zeta)} \right| \right\}, \\ \mathcal{U}_2^{(j)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j), \frac{2}{3} \left| \frac{f_2(\zeta)\rho(z_j)^{i_{2,j}}}{P_{2,j}(\zeta)} \right| > \left| \frac{f_1(\zeta)\rho(z_j)^{i_{1,j}}}{P_{1,j}(\zeta)} \right| \right\}.\end{aligned}$$

We still denote by χ a smooth function on $\mathbb{C}^2 \setminus \{0\}$ such that $\chi(z_1, z_2) = 1$ if $|z_1| > \frac{2}{3}|z_2|$ and $\chi(z_1, z_2) = 0$ if $|z_1| < \frac{1}{3}|z_2|$; and we set $\chi_1^{(j)}(\zeta) = \chi\left(\frac{f_1(\zeta)\rho(z_j)^{i_{1,j}}}{P_1^{(j)}(\zeta)}, \frac{f_2(\zeta)\rho(z_j)^{i_{2,j}}}{P_2^{(j)}(\zeta)}\right)$, $\chi_2^{(j)}(\zeta) = 1 - \chi_1^{(j)}(\zeta)$ and

$$\begin{aligned}\hat{g}_1^{(j)}(z) &= \frac{1}{Q_1^{(j)}(z)} \left(\tilde{g}_1^{(j)}(z) + \chi_1^{(j)}(z) P_{2,j}(z) e^{(j)}(z) \right), \\ \hat{g}_2^{(j)}(z) &= \frac{1}{Q_2^{(j)}(z)} \left(\tilde{g}_2^{(j)}(z) + \chi_2^{(j)}(z) P_{1,j}(z) e^{(j)}(z) \right).\end{aligned}$$

Therefore $g = \hat{g}_1^{(j)} f_1 + \hat{g}_2^{(j)} f_2$ on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ and in order to apply Theorem 6.1, the assumptions (b) and (c) are left to be shown.

From Lemma 5.1, for all $j \in \mathbb{N}$ and all $z \in \mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$, we have

$$\left| \frac{1}{Q_{2,j}(z)} \tilde{g}_2^{(j)}(z) \right| \lesssim \sum_{k=1}^{i_{2,j}} |\rho(z_j)|^{\frac{k-1}{2}} \left| g_{z_j+z_1^* \eta_{z_j}, v_{z_j}}^{(2)}[\alpha_{1,1}(z_1^*), \dots, \alpha_{1,k}(z_1^*)] \right|$$

uniformly with respect to z and j .

Therefore

$$(23) \quad \sum_{j \in \mathbb{N}} \int_{\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)} \left| \frac{1}{Q_{2,j}(z)} \tilde{g}_2^{(j)}(z) \right|^q dV(z) \lesssim c_{q,\kappa,(z_j)}^{(l)}(g)$$

and in particular $\frac{1}{Q_{2,j}} \tilde{g}_2^{(j)}$ is an holomorphic function with L^q -norm on $\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$ lower than $(c_{q,\kappa,(z_j)}^{(2)}(g))^{\frac{1}{q}}$. Thus Cauchy's inequalities imply that for all $\alpha, \beta \in \mathbb{N}$ and all $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ that

$$(24) \quad \left| \frac{\partial^{\alpha+\beta}}{\partial z_1^{*\alpha} \partial z_2^{*\beta}} \left(\frac{1}{Q_{2,j}} \tilde{g}_2^{(j)}(z) \right) \right| \lesssim c_{q,\kappa,(z_j)}^{(l)}(g) |\rho(z_j)|^{-\alpha - \frac{\beta}{2}}.$$

Since $\frac{g}{\max(|f_1|, |f_2|)}$ belongs to $L^q(D)$, g itself belongs to $L^q(D)$ and so

$$\int_{\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)} |e^{(j)}(z)|^q dV(z) \lesssim |\rho(z_j)|^{-q \frac{i_{1,j} + i_{2,j}}{2}} \int_{\mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)} |g(z)|^q dV(z).$$

In particular, for all α and β and all $z \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, we have

$$(25) \quad \left| \frac{\partial^{\alpha+\beta} e^{(j)}}{\partial z_1^{*\alpha} \partial z_2^{*\beta}}(z) \right| \lesssim |\rho(z_j)|^{-q \frac{i_{1,j} + i_{2,j}}{2} - \alpha - \frac{\beta}{2}}.$$

The inequalities (24) and (25) imply that the hypothesis (c) of Theorem 6.1 is satisfied by $\hat{g}_2^{(j)}$ for some large N , the same is also true for $\hat{g}_1^{(j)}$.

Now, on $\mathcal{U}_2^{(j)}$, we have by Lemma 5.1 :

$$\left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right| \lesssim \frac{1}{|\rho(z_j)|} \int_{(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}} \frac{|g(\zeta_1^*, \xi)|}{\max(|f_1(\zeta_1^*, \xi)|, |f_2(\zeta_1^*, \xi)|)} dV(\xi)$$

and so

$$\int_{\mathcal{U}_2 \cap \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)} \left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right|^q dV(z) \lesssim \int_{\mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)} \left(\frac{|g(\zeta_1^*, \xi)|}{\max(|f_1(\zeta_1^*, \xi)|, |f_2(\zeta_1^*, \xi)|)} \right)^q dV(\xi)$$

and since $(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j))_{j \in \mathbb{N}}$ is a κ -covering :

$$(26) \quad \sum_{j \in \mathbb{N}} \int_{\mathcal{U}_2 \cap \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)} \left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right|^q dV(z) \lesssim \left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^q(D)}^q.$$

Since for all $\alpha, \beta \in \mathbb{N}$, $\left| \frac{\partial^{\alpha+\beta} \chi_2^{(j)}}{\partial \zeta_1^{\alpha} \partial \bar{\zeta}_2^{\beta}}(z) \right| \lesssim |\rho(z_j)|^{-\alpha-\frac{\beta}{2}}$, (26) and (23) imply that $(\hat{g}_2^{(j)})_{j \in \mathbb{N}}$ satisfy the assumption (b) of Theorem 6.1 that we can therefore apply. \square

REFERENCES

- [1] W. Alexandre, E. Mazzilli : *Extension with growth estimates of holomorphic functions defined on singular analytic spaces*, arXiv:1101.4200.
- [2] E. Amar : *On the corona problem*, J. Geom. Anal. 1 (1991), no. 4, 291305.
- [3] E. Amar, J. Bruna : *On H^p -solutions of the Bezout equation in the ball*, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl. 1995, Special Issue, 715.
- [4] M. Andersson, H. Carlsson: *Wolf type estimates and the H^p corona problem in strictly pseudoconvex domains*, Ark. Mat. 32 (1994), no. 2, 255–276.
- [5] M. Andersson, H. Carlsson: *H^p -estimates of holomorphic division formulas*, Pacific J. Math. 173 (1996), no. 2, 307–335.
- [6] M. Andersson, H. Carlsson: *Estimates of solutions of the H^p and BMOA corona problem*, Math. Ann. 316 (2000), no. 1, 83–102.
- [7] P. Bonneau, A. Cumenge, A. Zeriahi : *Division dans les espaces de Lipschitz de fonctions holomorphes*, Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, 7387, Lecture Notes in Math., 1198, Springer, Berlin, 1986.
- [8] L. Carleson : *Interpolations by bounded analytic functions and the corona problem*, Ann of Math. (2) 76 1962 547559.
- [9] B. Berndtsson, M. Andersson: *Henkin-Ramirez formulas with weight factors*, Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, v-vi, 91-110.
- [10] B. Berndtsson: *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman*, Ann. Inst. Fourier, 4 (1996), 1083-1094.
- [11] J. Bruna, P. Charpentier, Y. Dupain: *Zero varieties for the Nevanlinna class in convex domains of finite type in \mathbb{C}^n* , Ann. Math. 147 (1998), 391-415.
- [12] S.G. Krantz, Song-Ying Li : *Some remarks on the corona problem on strongly pseudoconvex domains in \mathbb{C}^n* , Illinois J. Math. 39 (1995), no. 2, 323349.
- [13] E. Mazzilli: *Formules de division dans \mathbb{C}^n* , Michigan Math. J. 51 (2003), no. 2, 251277.
- [14] E. Mazzilli: *Division des distributions et applications à l'étude d'idéaux de fonctions holomorphes*, C.R. Acad. Sci. Paris, Ser. I 338 (2004), 1-6.
- [15] E. Mazzilli: *Courants du type résiduel attachés à une intersection complète*, J. Math. Anal. Appl., 368 (2010), 169-177.
- [16]

- [17] P. Montel: *Sur une formule de Darboux et les polynômes d'interpolation*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Sér. 2, 1 no. 4 (1932), p. 371-384. J. Ortega, J. Fàbrega : *Multipliers in Hardy-Sobolev spaces*, Integral Equations Operator Theory 55 (2006), no. 4, 535-560.
- [18] H. Skoda : *Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Sci. cole Norm. Sup. (4) 5 (1972), 545-579.
- [19] N. Varopoulos: *BMO functions and the $\bar{\partial}$ -equation*, Pacific J. Math. 71 (1977), no. 1, 221-273.

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